On Pseudo Symmetric Ideals in Γ-Semirings

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Abstract

We introduce the notions of pseudo symmetric ideals in Γ-semiring and pseudo symmetric Γ-Semirings. We characterize pseudo symmetric ideals in Γ-Semiring and exhibit some examples and some classes of pseudo symmetric Γ-Semirings.

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1 Introduction

The notion of Γ-semiring was introduced by M.Muralikrishna Rao as a generalization of Γ-ring, rings and semiring. All definitions and fundamental concepts concerning Γ-semirings can be found in [6],[7],[8],[9] and references therein. In [1],[2] and [3], A.Aanjeyulu initiated the study of pseudo symmetric ideals in semigroups and pseudo symmetric semigroups. In this paper, we introduce the notions of pseudo symmetric ideals in Γ-semiring, pseudo symmetric Γ-semirings and characterize pseudo symmetric ideals in Γ-semiring.

2 Preliminaries

Definition 2.1. Let \(S\) and \(\Gamma\) be two additive semigroups. Then \(S\) is called Γ-semiring if there exists a mapping \(S \times \Gamma \times S \to S\) (image to be denoted by \(aab\) for \(a, b \in S, \alpha \in \Gamma\)) satisfying the following conditions.
\begin{itemize}
(i) \(a\alpha(b + c) = a\alpha b + a\alpha c\)
(ii) \((a + b)\alpha c = a\alpha c + b\alpha c\)
(iii) \(a(\alpha + \beta)b = a\alpha b + a\beta b\)
(iv) \(a\alpha(b\beta c) = (a\alpha b)\beta c\), for all \(a, b, c \in S\) and for all \(\alpha, \beta \in \Gamma\).
\end{itemize}

It is obvious that every semiring \(S\) is a \(\Gamma\)-semiring where \(\Gamma = S\) and \(x\alpha y\) denotes the product of the elements \(x, y, \alpha \in S\). Every \(\Gamma\)-ring is also a \(\Gamma\)-semiring.

A \(\Gamma\)-semiring \(S\) is said to be commutative if \(a\alpha b = b\alpha a\), for all \(a, b \in S\) and for all \(\alpha \in \Gamma\).

Let \(S\) be a \(\Gamma\)-semiring and let \(A\) be a non empty subset of \(S\). \(A\) is called a sub \(\Gamma\)-semiring of \(S\) if \(A\) is a sub semigroup of \((S, +)\) and \(A\Gamma A \subseteq A\).

**Example 2.2.** Let \(S\) be an additive semigroup of all \(m \times n\) matrices over the set of all non negative rational numbers and let \(\Gamma\) be the additive semigroup of all \(n \times m\) matrices over the set of all non-negative integers. Then \(S\) is a \(\Gamma\)-semiring with matrix multiplication as the ternary operation.

**Example 2.3.** Let \(S\) be an arbitrary semiring and let \(\Gamma\) be a semigroup. Define a mapping \(S \times \Gamma \times S \to S\) by \(aab = ab\), for all \(a, b \in S\) and \(\alpha \in \Gamma\). It is easy to see that \(S\) is a \(\Gamma\)-semiring. Thus a semiring can be considered to be a \(\Gamma\)-semiring. Since every ring is a semiring, a ring can be considered to be a \(\Gamma\)-semiring.

**Definition 2.4.** An ideal \(A\) of a \(\Gamma\)-semiring \(S\) is called a prime ideal provided \(X\Gamma Y \subseteq A\); \(X, Y\) are ideals of \(S\), then either \(X \subseteq A\) or \(Y \subseteq A\) or \(x\Gamma S\Gamma y \subseteq A\) implies either \(x \in A\) or \(y \in A\).

**Definition 2.5.** An ideal \(A\) of a \(\Gamma\)-semiring \(S\) is called a completely prime ideal provided \(x\Gamma y \subseteq A\); \(x, y \in S\) implies either \(x \in A\) or \(y \in A\).

**Definition 2.6.** An ideal \(A\) of a \(\Gamma\)-semiring \(S\) is called a semiprime ideal provided \(x\Gamma S\Gamma x \subseteq A\); \(x \in S\) implies \(x \in A\).

**Definition 2.7.** An ideal \(A\) of a \(\Gamma\)-semiring \(S\) is called a completely semiprime ideal provided \((x\Gamma)^{n-1} x \subseteq A\); \(x \in S\) for some natural number \(n\) implies \(x \in A\).

**Definition 2.8.** A \(\Gamma\)-semiring \(S\) is said to be a left(right) duo \(\Gamma\)-semiring provided every left(right) ideal of \(S\) is a two sided ideal of \(S\).

**Definition 2.9.** A \(\Gamma\)-semiring \(S\) is said to be a duo \(\Gamma\)-semiring provided it is both a left and a right duo \(\Gamma\)-semiring.

**Definition 2.10.** A \(\Gamma\)-semiring \(S\) is said to be left(right) pseudo commutative provided \(a\Gamma b\Gamma c = b\Gamma a\Gamma c\) for all \(a, b, c \in S\).
Definition 2.11. A $\Gamma$-semiring $S$ is said to be quasi commutative provided for any $a, b \in S$, there exists a natural number $n$ such that $a\Gamma b = (b\Gamma)^{n-1}b\Gamma a$.

Definition 2.12. A $\Gamma$-semiring $S$ is said to be a generalised commutative $\Gamma$-semiring provided $S$ contains $1$ as an $r$-element.

Definition 2.13. An element $'a'$ of a $\Gamma$-semiring $S$ is said to be an $r$-element provided $a\Gamma s = s\Gamma a$, for all $s \in S$ and if $x, y \in S$, we have $a\Gamma x\Gamma y = b\Gamma y\Gamma x$, for some $b \in S$.

Definition 2.14. A $\Gamma$-semiring $S$ is said to be normal provided $a\Gamma S = S\Gamma a$, for all $a \in S$.

Definition 2.15. A $\Gamma$-semiring $S$ is said to be an idempotent $\Gamma$-semiring provided every element is an idempotent.

Definition 2.16. An element $'a'$ of a $\Gamma$-semiring $S$ is said to be an idempotent provided $a\alpha a = a$, for all $\alpha \in \Gamma$.

Definition 2.17. An element $'a'$ of a $\Gamma$-semiring $S$ is said to be a mid unit provided $x\Gamma a\Gamma y = x\Gamma y$ for any $x, y \in S$.

Definition 2.18. An ideal $A$ of a $\Gamma$-semiring $S$ is said to be left(right) primary provided that the following conditions hold:

(i) If $X, Y$ are ideals of $S$ such that $XTY \subseteq A$ and $Y \not\subseteq A$ ($X \not\subseteq A$), then $X \subseteq M_A(S)$ ($Y \subseteq M_A(S)$), $M_A(S)$ is the intersection of all prime ideals of $S$ containing $A$.

(ii) $M_A(S)$ is a prime ideal of $S$.

Definition 2.19. Let $S$ be a $\Gamma$-semiring with a zero element $0$ and let $A$ be an ideal of $S$. Then $A$ is said to be nilpotent if $(A\Gamma)^{n-1}A = 0$, for some integer $n > 0$.

3 Pseudo Symmetric Ideals

Definition 3.1. An ideal $A$ in a $\Gamma$-semiring $S$ is said to be pseudo symmetric provided $x\Gamma y \subseteq A$; $x, y \in S$ implies $x\Gamma s\Gamma y \subseteq A$, for all $s \in S$.

Definition 3.2. A $\Gamma$-semiring $S$ is said to be pseudo symmetric provided every ideal is pseudo symmetric ideal.

We remark that every commutative $\Gamma$-semiring is a pseudo symmetric $\Gamma$-semiring and the converse need not necessarily be true.
Example 3.3. Let \( S = \{a, b, c\} \) and \( \Gamma = \{a, b, c\} \). We define binary operations \( \cdot \)' in \( S \) and \( \ast \)' in \( \Gamma \) as shown in the following tables:

\[
\begin{array}{|c|c|c|c|}
\hline
\cdot & a & b & c \\
\hline
a & a & a & a \\
b & a & a & a \\
c & a & b & c \\
\hline
\end{array}
\quad \quad \quad
\begin{array}{|c|c|c|c|}
\hline
\ast & a & b & c \\
\hline
a & a & b & a \\
b & a & b & a \\
c & a & b & c \\
\hline
\end{array}
\]

Now, clearly \( (S, \cdot) \) and \( (\Gamma, \ast) \) are semigroups. Define a mapping \( S \times \Gamma \times S \rightarrow S \) by \( a \cdot b = ab \), for all \( a, b \in S \) and \( \alpha \in \Gamma \). It is easy to see that \( S \) is a \( \Gamma \)-semiring. Now clearly \( S \) is pseudo symmetric \( \Gamma \)-semiring which is not commutative \( \Gamma \)-semiring.

We now discuss the relationships among prime, completely prime and pseudo symmetric ideals.

Theorem 3.4. Let \( S \) be a \( \Gamma \)-semiring. Then the following statements hold:

(i) Every completely prime ideal is both prime and pseudo symmetric.

(ii) Let \( A \) be a pseudo symmetric ideal of \( S \). Then \( A \) is prime \( \iff \) \( A \) is completely prime.

(iii) Let \( A \) be a prime ideal of \( S \). Then \( A \) is pseudo symmetric \( \iff \) \( A \) is completely prime.

Proof.

(i) This statement is easy to observe. We hence omit the proof.

(ii) This result follows from lemma 1 in [2].

(iii) Let \( A \) be a pseudo symmetric ideal of \( S \). If \( x \Gamma y \subseteq A \), for some \( x, y \in S \), then \( x \Gamma S \Gamma y \subseteq A \). Since \( A \) is prime, we have \( x \in A \) or \( y \in A \). This shows that \( A \) is completely prime. Conversely if \( A \) completely prime ideal, then from (i) \( A \) is pseudo symmetric ideal.

Theorem 3.5. The following statements are equivalent for an ideal \( A \) in a \( \Gamma \)-semiring \( S \).

(i) \( A \) is a pseudo symmetric ideal.

(ii) \( A_r(a) = \{x \in S; a \Gamma x \subseteq A\} \) is an ideal of \( S \) for all \( a \in S \).

(iii) \( A_l(a) = \{x \in S; x \Gamma a \subseteq A\} \) is an ideal of \( S \) for all \( a \in S \).

Proof.

(i)\( \Rightarrow \) (ii) :

Let \( x \in A_r(a) \). Then \( a \Gamma x \subseteq A \). Since \( A \) is a pseudo symmetric ideal, \( a \Gamma s \Gamma x \subseteq A \) and clearly \( a \Gamma x \Gamma s \subseteq A \). Therefore \( s \Gamma x, x \Gamma s \subseteq A_r(a) \) and hence \( A_r(a) \) is an ideal in \( S \).

(ii)\( \Rightarrow \) (i) :
Let \( x\Gamma y \subseteq A \). Then \( y \in A_r(x) \). Since \( A_r(x) \) is an ideal, we have \( s\Gamma y \subseteq A_r(x) \) and hence \( x\Gamma s\Gamma y \subseteq A \), for all \( s \in S \). Therefore \( A \) is a pseudo symmetric ideal.

(i) \( \Rightarrow \) (iii) :

Let \( x \in A_l(a) \). Therefore \( x\Gamma a \subseteq A \). Since \( A \) is a pseudo symmetric ideal, \( x\Gamma s\Gamma a \subseteq A \) and clearly \( s\Gamma x\Gamma a \subseteq A \). Therefore \( x\Gamma s, s\Gamma x \subseteq A_l(a) \) and hence \( A_l(a) \) is an ideal in \( S \).

(iii) \( \Rightarrow \) (i) :

Let \( x\Gamma y \subseteq A \). Then \( x \in A_l(y) \). Since \( A_l(y) \) is an ideal of \( S \), \( x\Gamma s \subseteq A_l(y) \), for all \( s \in S \) and hence \( x\Gamma s\Gamma y \subseteq A \), for all \( s \in S \). Therefore \( A \) is a pseudo symmetric ideal of \( S \).

We now describe the relationship between the one-side duo \( \Gamma \)-semiring and the pseudo symmetric \( \Gamma \)-semiring.

**Corollary 3.6.** Every left(right) duo \( \Gamma \)-semiring \( S \) is a pseudo symmetric \( \Gamma \)-semiring.

**Proof.** Let \( A \) be any ideal in \( S \). Since for all \( a \in S \), \( A_l(a) \) is a left ideal and hence by theorem 3.5, \( A \) is a pseudo symmetric ideal. Therefore \( S \) is a pseudo symmetric \( \Gamma \)-semiring. Similarly every right duo \( \Gamma \)-semiring is a pseudo symmetric \( \Gamma \)-semiring. \( \square \)

**Remark 3.7.** In fact pseudo symmetric \( \Gamma \)-semirings are in abundance.

1. Every left(right) pseudo commutative \( \Gamma \)-semiring is a pseudo symmetric \( \Gamma \)-semiring.
2. Every quasi commutative \( \Gamma \)-semiring is a pseudo symmetric \( \Gamma \)-semiring.
3. Every generalized commutative \( \Gamma \)-semiring is a pseudo symmetric \( \Gamma \)-semiring.
4. Every normal \( \Gamma \)-semiring is a pseudo symmetric \( \Gamma \)-semiring.

**Lemma 3.8.** Every idempotent \( \Gamma \)-semiring \( S \) is a pseudo symmetric \( \Gamma \)-semiring.

**Proof.** Let \( A \) be any ideal in \( S \) and let \( a\Gamma b \subseteq A \). Then \( b\Gamma a = b\Gamma a\Gamma b\Gamma a \subseteq A \) and hence \( a\Gamma s\Gamma b = a\Gamma s\Gamma b\Gamma a\Gamma s\Gamma b \subseteq A \). Therefore \( A \) is a pseudo symmetric ideal. \( \square \)

**Lemma 3.9.** If \( S \) is a \( \Gamma \)-semiring in which every element is a mid unit, then \( S \) is a pseudo symmetric \( \Gamma \)-semiring.
Proof. Let \( A \) be an ideal in \( S \) and let \( a \Gamma b \subseteq A \). Now, for any \( s \in S \), \( a \Gamma s \Gamma b = a \Gamma b \subseteq A \). So, \( A \) is a pseudo symmetric ideal.

\[ \square \]

**Lemma 3.10.** Every completely semiprime ideal \( A \) in a \( \Gamma \)-semiring \( S \) is a pseudo symmetric ideal and the converse is not true.

**Proof.** Let \( x \Gamma y \subseteq A \). Then \((y \Gamma x \Gamma)^{1}y \Gamma x = y \Gamma x \Gamma y \Gamma x \subseteq A \). Since \( A \) is a completely semiprime ideal, \( y \Gamma x \subseteq A \). Now, \((x \Gamma s \Gamma y \Gamma)^{1}x \Gamma s \Gamma y = x \Gamma s \Gamma y \Gamma x \Gamma s \Gamma y \subseteq A \), for all \( s \in S \) and hence \( x \Gamma s \Gamma y \subseteq A \). Therefore \( A \) is a pseudo symmetric ideal.

\[ \square \]

The following theorem gives a characterization for a pseudo symmetric ideal to be a one sided primary ideal.

**Theorem 3.11.** Let \( S \) be a \( \Gamma \)-semiring and \( A \) a pseudo symmetric ideal of \( S \). Then \( A \) is left primary if and only if the following condition holds:

For all \( x, y \in S \), \( x \Gamma y \subseteq A \) and \( y \notin A \) imply \((x \Gamma)^{n-1}x \subseteq A \), for some \( n > 0 \) ——— (\(*\))

**Proof.** Suppose that \( A \) is a left primary ideal of \( S \) and \( x \Gamma y \subseteq A \) with \( y \notin A \). Then, since \( A \) is pseudo symmetric and \( x \Gamma y \subseteq A \), we have \((x \Gamma)^{n-1}x \subseteq A \). Then, we have \((x \Gamma)^{n-1}_x \subseteq A \), where \( M_A(S) \) is the intersection of all prime ideals of \( S \) containing \( A \). Then we have \( M_A(S) = N_A(S) \), where \( N_A(S) \) is the set of all elements of \( S \) nilpotent with respect to \( A \). This implies that \( x \in N_A(S) \) and so \((x \Gamma)^{n-1}x \subseteq A \), for some \( n \geq 1 \). Hence (\(*\)) holds.

Suppose that (\(*\)) holds. Then we have the following situations:

(i) \( X \) and \( Y \) are ideals of \( S \) with \( XY \subseteq A \) but \( Y \notin A \). Then there exists an element \( y \in Y \) but \( y \notin A \) such that for all \( x \in X \), \( x \Gamma y \subseteq XY \subseteq A \). By (\(*\)), we immediately obtain that \( x \in M_A(S) \), for all \( x \in X \). This implies that \( X \subseteq M_A(S) \).

(ii) Assume that \( x \Gamma y \subseteq M_A(S) \). Then we have \( x \Gamma y \subseteq N_A(S) \) and hence we find a smallest positive integer \( n \) such that \((x \Gamma)^{n-1}x \Gamma y \subseteq A \). If \( n = 1 \), then \( x \Gamma y \subseteq A \). By (\(*\)), we have \((x \Gamma)^{k-1}x \subseteq A \), for some integer \( k > 0 \) or \( y \in A \). This means that \( x \in N_A(S) \) or \( y \in M_A(S) \). Now, we assume that \( n > 1 \). We have the following cases:

Case (i). If \( y \Gamma (x \Gamma y \Gamma)^{n-2}x \Gamma y \notin A \), then by (\(*\)) and \( x \Gamma (y \Gamma (x \Gamma y \Gamma)^{n-2}x \Gamma y) = (x \Gamma y \Gamma)^{n-1}x \Gamma y \subseteq A \), we have \((x \Gamma)^{n-1}x \subseteq A \), for some \( n > 0 \). This implies that \( x \in N_A(S) = M_A(S) \).

Case (ii). If \( y \Gamma (x \Gamma y \Gamma)^{n-2}x \Gamma y \subseteq A \), then since \((x \Gamma y \Gamma)^{n-2}x \Gamma y \notin A \), we have \( y \in N_A(S) = M_A(S) \),

Hence, in all cases we must have \( x \in M_A(S) \) or \( y \in M_A(S) \). This shows that \( M_A(S) \) is completely prime, and so \( M_A(S) \) is prime.

\[ \square \]
Corollary 3.12. If $S$ is a one-side duo $\Gamma$-semiring, then an ideal $A$ of $S$ is left primary if and only if $(\ast)$ holds.

We now make an attempt to characterize pseudo symmetric ideals in $\Gamma$-semiring.

Lemma 3.13. Let $A$ be any pseudo symmetric ideal in a $\Gamma$-semiring $S$. Then $a_1a_2a_3\ldots a_{n-1}a_n \in A$ if and only if $\langle a_1 \rangle \cap \langle a_2 \rangle \cap \ldots \cap \langle a_n \rangle \subseteq A$.

Proof. Let $A$ be any pseudo symmetric ideal in a $\Gamma$-semiring $S$. Clearly if $\langle a_1 \rangle \cap \langle a_2 \rangle \cap \ldots \cap \langle a_n \rangle \subseteq A$, then $a_1a_2a_3\ldots a_{n-1}a_n \in A$. Conversely if $a_1a_2a_3\ldots a_{n-1}a_n \in A$, then for any $t \in \langle a_1 \rangle \cap \langle a_2 \rangle \cap \ldots \cap \langle a_n \rangle$, we have $t = s_1a_1s_2a_2s_3a_3\ldots a_{n-1}a_n$, where $s_i \in S^1$ and $\alpha_i, \beta_i \in \Gamma$. Since $A$ is a pseudo symmetric ideal, we have $t \in A$. Therefore $\langle a_1 \rangle \cap \langle a_2 \rangle \cap \ldots \cap \langle a_n \rangle \subseteq A$. □

Remark 3.14. Let $S$ be a $\Gamma$-semiring. An element $a \in S$ is called a left identity (resp. right identity) of $S$ if $x = a\alpha x$ (resp. $x = x\alpha a$) for all $x \in S$ and $\alpha \in \Gamma$. If $'a'$ is both a left and right identity, then $'a'$ is called an identity of $S$.

Let $S$ be a $\Gamma$-semiring. If $S$ has an identity element 1, set $S^1 = S$ and if $S$ does not have an identity element 1, let $S^1$ be the $\Gamma$-semiring $S$ with an identity element 1 adjoined.

Corollary 3.15. If $A$ is a pseudo symmetric ideal in a $\Gamma$-semiring $S$, then for any natural number $n$, $(aa)^{n-1}a \in A$; $\alpha \in \Gamma$ implies $(\langle \alpha \rangle \Gamma)^{n-1} \subseteq A$.

Proof. The proof of this corollary follows from lemma 3.13 by taking $a_1 = a_2 = a_3 = \ldots = a_n = a$. □

Theorem 3.16. Every prime ideal $P$ minimal relative to containing a pseudo symmetric ideal $A$ in a $\Gamma$-semiring $S$ is completely prime.

Proof. Let $T$ be the sub $\Gamma$-semiring generated by $S \setminus P$. First we show that $A \cap T = \phi$. If $A \cap T \neq \phi$, then there exist $x_1, x_2, x_3, \ldots, x_n \in S \setminus P$ such that $x_1\alpha_1x_2\ldots x_{n-1}x_n \in A$. By lemma 3.13, we have $\langle x_1 \rangle \cap \langle x_2 \rangle \cap \ldots \cap \langle x_n \rangle \subseteq A \subseteq P$. Since $P$ is a prime ideal, we have $\langle x_i \rangle \subseteq P$ for some $i$, a contradiction. Thus $A \cap T = \phi$. Consider the set $\Sigma = \{B; \ B$ is an ideal in $S$ containing $A$ such that $B \cap T = \phi\}$. Since $A \in \Sigma$, $\Sigma$ is not empty. Now, $\Sigma$ is a poset under set inclusion and satisfies the hypothesis of Zorn’s lemma. Thus by Zorn’s lemma, $\Sigma$ contains a maximal element, say $M$. Let $X$ and $Y$ be two ideals in $S$ such that $XY \subseteq M$. If $X \notin M$ and $Y \notin M$, then $M \cup X$ and $M \cup Y$ are ideals in $S$ containing $M$ properly and hence by the maximality of $M$, we have $(M \cup X) \cap T \neq \phi$ and $(M \cup Y) \cap T \neq \phi$. Since $M \cap T = \phi$, we have $X \cap T \neq \phi$ and $Y \cap T \neq \phi$. So there exists $x \in X \cap T$ and $y \in Y \cap T$. Now,
$x \Gamma y \subseteq X Y \cap T \subseteq M \cap T = \phi$, a contradiction. Therefore $M$ is a prime ideal containing $A$. Now, $A \subseteq M \subseteq S \setminus T \subseteq P$. Since $P$ is a minimal prime ideal relative to containing $A$, we have $M = S \setminus T = P$. Therefore $P$ is a completely prime ideal.

\begin{proof}
By lemma 3.10, $A$ is a pseudo symmetric ideal and hence by theorem 3.16, $P$ is a completely prime ideal.
\end{proof}

\textbf{Corollary 3.17.} Every prime ideal $P$ minimal relative to containing a completely semi prime ideal $A$ in a $\Gamma$-semiring $S$ is completely prime.

\textit{Proof.} By lemma 3.10, $A$ is a pseudo symmetric ideal and hence by theorem 3.16, $P$ is a completely prime ideal.

\textbf{References}


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