Some Change of Rings Results for Gorenstein Flat Dimension

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Abstract

In this paper, we computed the Gorenstein weak dimension of GF-closed polynomial rings, and we establish a general change of rings result for Gorenstein flat dimension.

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1 Introduction

Throughout the paper all commutative rings with identity, and all modules are unitary.

Let $R$ be a ring and let $M$ be an $R$-module. The flat dimension of $M$ is denoted by $\text{fd}_R(M)$. For the polynomial ring $R[X]$ in one indeterminate $X$ over $R$, we use $M[X]$ to denote the $R[X]$-module $M \otimes_R R[X]$.

We say that $M$ is Gorenstein flat, if there exists an exact sequence of flat $R$-modules, $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$, such that $M \cong \text{Im}(F_0 \to F^0)$ and such that $I \otimes_R -$ leaves the sequence exact whenever $I$ is an injective $R$-module.

For a positive integer $n$, we say that $M$ has Gorenstein flat dimension at most $n$, and we write $\text{Gfd}_R(M) \leq n$, if there is an exact sequence of $R$-modules $0 \to G_n \to \cdots \to G_0 \to M \to 0$, where each $G_i$ is Gorenstein flat (see [8, 10, 12]).

The notion of Gorenstein flat modules was introduced and studied over Gorenstein rings by Enochs, Jenda, and Torrecillas [11], as a generalization of the notion of flat modules in the sense that an $R$-module is flat if and
only if it is Gorenstein flat with finite flat dimension (this is proved to hold over any associative ring [2, Theorem 2.2]). In [7], Chen and Ding generalized known characterizations of Gorenstein flat modules (then of the Gorenstein flat dimension) over Gorenstein rings to $n$-FC rings (coherent with finite self-FP-injective dimension). And in [12], Holm generalized the study of the Gorenstein flat dimension to coherent rings. In the same direction, Bennis [1] generalized the study of Gorenstein flat dimension to a larger class of rings called GF-closed. Recall that a ring $R$ is called GF-closed, if the class of all Gorenstein flat $R$-modules is closed under extensions; that is: if for every short exact sequence of $R$-modules $0 \to A \to B \to C \to 0$, the condition $A$ and $C$ are Gorenstein flat implies that $B$ is Gorenstein flat. The class of GF-closed rings includes strictly the one of coherent rings and also the one of rings of finite weak dimension [1, Example 3.6].

In this paper, we give two change of rings results for Gorenstein flat dimension. The first one (Theorem 2.1), which generalizes [4, Theorem 2.10], computes the Gorenstein weak dimension of GF-closed polynomial rings, such that the Gorenstein weak dimension of a ring $R$, $\text{Gw}(R)$, is defined by:

$$\text{Gw}(R) = \sup \{ \text{Gf}(M) \mid M \text{ is an } R\text{-module} \}.$$ 

In the classical homological dimension theory the following result is well-known: for a ring homomorphism $R \to S$ and an $S$-module $M$, we have $\text{fd}_R(M) \leq \text{fd}_S(M) + \text{fd}_R(S)$ (see for instance [6, Exercise 5, p. 360]). Naturally one would like to establish the Gorenstein counterpart of this result. In the second main result of this paper (Theorem 2.5), we establish this result under a condition.

2 Main Results

Recall the Gorenstein weak dimension of a ring $R$, $\text{Gw}(R)$, is defined by:

$$\text{Gw}(R) = \sup \{ \text{Gf}(M) \mid M \text{ is an } R\text{-module} \}.$$ 

In [4, Theorem 2.10], the Gorenstein weak dimension of coherent polynomial rings is computed. Here we generalize this result to GF-closed rings [1].

**Theorem 2.1** Let $R$ be a GF-closed commutative ring such that the polynomial ring $R[X]$ is also GF-closed. Then,

$$\text{Gw}(R[X]) = \text{Gw}(R) + 1.$$ 

To prove this theorem we need some results.

**Lemma 2.2** (Lemma 2.7(b), [9]) Let $R \to S$ be a ring homomorphism with $\text{fd}_R(S) < \infty$. The for an $R$-module $M$ and an $S$-module $F$, we have:

$$\text{Gf}(M \otimes_R F) \leq \text{fd}_S(F) + \text{Gf}(M).$$
Lemma 2.3 1. If $M$ is an $R[X]$-module such that the indeterminate $X$ is a non-zero-divisor on $M$, then $\text{Gfd}_R(M/XM) \leq \text{Gfd}_{R[X]}(M)$.

2. For an $R$-module $M$, we have $\text{Gfd}_{R[X]}(M[X]) \leq \text{Gfd}_R(M)$.

Proof. 1. The inequality is a particular case of Lemma 2.2.

2. Also follows from Lemma 2.2 since $R[X]$ is a free $R$-module.

If we consider GF-closed rings, we get:

Lemma 2.4 If the polynomial ring $R[X]$ over a ring $R$ is GF-closed, then, for every $R[X]$-module $M$, we have $\text{Gfd}_{R[X]}(M) \leq \text{Gfd}_R(M) + 1$.

If furthermore $R$ is GF-closed and $XM = 0$, then $\text{Gfd}_{R[X]}(M) = \text{Gfd}_R(M) + 1$.

Proof. Consider the short exact sequence of $R[X]$-modules [13, Lemma 9.29]:

$$0 \longrightarrow M[X] \longrightarrow M[X] \longrightarrow M \longrightarrow 0.$$  

From [1, Theorem 2.11 (4)], $\text{Gfd}_{R[X]}(M) \leq \text{Gfd}_{R[X]}(M[X]) + 1$. Therefore, using the inequality of Lemma 2.3(2), we get $\text{Gfd}_{R[X]}(M) \leq \text{Gfd}_R(M) + 1$.

Now, suppose that $XM = 0$. To prove the equality, it remains, from the first inequality, to prove the converse inequality $\text{Gfd}_R(M) \leq \text{Gfd}_{R[X]}(M) - 1$. For that we argue similarly to the proof of [5, Lemma 3 (c)]. We can assume that $\text{Gfd}_{R[X]}(M) = n$ for some positive integer $n$. If $n = 0$ (that is $M$ is a Gorenstein flat $R[X]$-module), then it embeds in a flat $R[X]$-module. But, from [5, Example (7), p. 9], this contradicts the fact that $XM = 0$ and so $n > 0$. Then, there exists a short exact sequence of $R[X]$-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where $F$ is free and $\text{Gfd}_{R[X]}(K) = n - 1$. Applying the functor $- \otimes_{R[X]} R$, where $R[X]/XR[X] = R$, to the above short exact sequence and using [5, Examples (1), p. 102], we get the following sequence of $R$-modules

$$0 \longrightarrow M \longrightarrow K/XK \longrightarrow F/XF \longrightarrow M \longrightarrow 0.$$  

From Lemma 2.3(1), $\text{Gfd}_R(K/XK) \leq \text{Gfd}_{R[X]}(K) = n - 1$. Then, since $F/XF$ is a free $R$-module and using [1, Theorem 2.11], we get $\text{Gfd}_R(M) \leq n - 1$, as desired.

Proof of Theorem 2.1. Let $M$ be an $R[X]$-module. From Lemma 2.4, we have $\text{Gfd}_{R[X]}(M) \leq \text{Gfd}_R(M) + 1 \leq \text{Gwdim}(R) + 1$. Then, $\text{Gwdim}(R[X]) \leq \text{Gwdim}(R) + 1$.

Conversely, consider an $R$-module $M$. Then, it is an $R[X]$-module satisfying $XM = 0$. Then, from Lemma 2.4, $\text{Gfd}_R(M) \leq \text{Gfd}_{R[X]}(M) - 1 \leq \text{Gwdim}(R[X]) - 1$. Therefore, $\text{Gwdim}(R) \leq \text{Gwdim}(R[X]) - 1$. □
Now, we give the following general change of rings result for Gorenstein flat dimension result. Compare this result to [6, Exercise 5, p. 360].

**Theorem 2.5** Let $R \rightarrow S$ be a ring homomorphism of GF-closed rings and let $M$ be an $S$-module. If every injective $R$-module has finite flat dimension, then

$$\text{Gfd}_R(M) \leq \text{Gfd}_S(M) + \text{fd}_R(S).$$

In the proof of this result we use the notion of strongly Gorenstein flat modules, which is introduced in [3] as follows:

**Definition 2.6 ([3], Definition 3.1)** An $R$-module $M$ is said to be strongly Gorenstein flat, if there exists an exact sequence of flat $R$-modules

$$F = \cdots \rightarrow F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

such that $M \cong \text{Im}(f)$ and such that $- \otimes_R I$ leaves the sequence $F$ exact whenever $I$ is an injective $R$-module.

Recall that every Gorenstein flat module is a direct summand of a strongly Gorenstein flat module [3, Theorem 3.5]. The important of this result manifests in the fact that the strongly Gorenstein flat modules have a simpler characterization [3, Proposition 3.6]. Here we generalize this characterization as follows:

**Lemma 2.7** An $R$-module $M$ is strongly Gorenstein flat if and only if there exists a short exact sequence of $R$-modules $0 \rightarrow M \rightarrow F \rightarrow M \rightarrow 0$, where $F$ is flat, and $\text{Tor}_i^R(M, I) = 0$ for some integer $i > 0$ and for any $R$-module $I$ with finite injective dimension (or for any injective $R$-module $I$).

**Proof.** The “only if” part follows from [3, Proposition 3.6]. We prove the “if” part. Note that if we apply $- \otimes_R I$ to the sequence $0 \rightarrow M \rightarrow F \rightarrow M \rightarrow 0$, we get

$$0 = \text{Tor}_{i+1}^R(F, I) \rightarrow \text{Tor}_{i+1}^R(M, I) \rightarrow \text{Tor}_i^R(M, I) \rightarrow \text{Tor}_i^R(F, I) = 0$$

Then, $\text{Tor}_{i+1}^R(M, I) \cong \text{Tor}_i^R(M, I)$. Thus, if $\text{Tor}_i^R(M, I) = 0$ for some integer $i > 0$, then $\text{Tor}_i^R(M, I) = 0$ for all $i > 0$. Therefore, $M$ is strongly Gorenstein flat (by [3, Proposition 3.6]).

**Proof of Theorem 2.5.** We can assume that $n = \text{fd}_R(S)$ is finite. By induction and from [1, Theorem 2.11], it suffices to prove the inequality for $\text{Gfd}_S(M) = 0$; that is $M$ is a Gorenstein flat $S$-module. Then, from [3, Theorem 3.5], $M$ is a direct summand of a strongly Gorenstein flat $S$-module $N$. 

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From Lemma 2.7, there is a short exact sequence of $S$-modules $0 \to N \to F \to N \to 0$, where $F$ is flat. By Horseshoe Lemma [13, Lemma 6.20], we get the following commutative:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & M_n & F_n & M_n & 0 \\
\downarrow & \downarrow & \downarrow & \\
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
0 & M_0 & M_0 \oplus M_0 & M_0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & M & F & M & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 \\
\end{array}
\]

where $M_i$ is a projective $R$-module for $i = 0, ..., n - 1$, and so $F_n$ is a flat $R$-module (since $\text{fd}_R(F) \leq \text{fd}_R(S)$ [6, Exercise 5, p. 360]). This implies, from Lemma 2.7, that $M_n$ is strongly Gorenstein flat (since, by hypothesis, for every injective $R$-module, there exists a positive integer $j$ such that $\text{Tor}^R_j(M_n, I) = 0$). Then, $\text{Gfd}_R(N) \leq \text{fd}_R(S)$. Therefore, from [1, Proposition 2.10], $\text{Gfd}_R(M) \leq \text{n} = \text{fd}_R(S)$. This completes the proof. \hfill \blacksquare

Finally, note that Iwanaga-Gorenstein rings (i.e., Noetherian rings with finite self-injective dimension) and rings of finite weak dimension are examples of rings that satisfying the condition on $R$ of Theorem 2.5.

References


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