

# A Note on the Closed Ideals of Prosemisimple Lie Algebras

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## Abstract

We extend the basic fact that every ideal of a finite dimensional semisimple Lie algebra has a unique complement to the case of closed ideals of prosemisimple Lie algebras. We prove that if  $A$  is a closed ideal of a prosemisimple Lie algebra  $L = \varprojlim L_n$  ( $n \in \mathbb{N}$ ), where the  $L_n$  are finite dimensional semisimple Lie algebras, then there exists a unique ideal  $B$  of  $L$  such that  $L = A \oplus B$ .

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## 1 Introduction

The notion of an inverse sequence and its limit were first discovered in 1929 by Alexandrov. But it was Lefschetz who laid most of the foundations of this subject in his famous book on Algebraic Topology in 1943 [7]. Subsequently, inverse limits began to be widely studied and applied in Topology and Algebra such as the inverse limits of linear algebraic groups (usually called the pro-affine algebraic groups) which appear naturally in the representation theory of Lie groups [1], [3], [6], [8], [9]. So it is of interest to make a generalization of the basic theory (found for example in [2] and [5]) concerning finite dimensional Lie algebras to the category of profinite dimensional Lie algebras. In this paper we extend the basic fact that every ideal of a finite dimensional semisimple Lie algebra has a unique complement to the case of closed ideals of prosemisimple Lie algebras. We prove that if  $A$  is a closed ideal of a prosemisimple Lie algebra  $L = \varprojlim L_n$  ( $n \in \mathbb{N}$ ), where the  $L_n$  are finite dimensional semisimple Lie algebras, then there exists a unique ideal  $B$  of  $L$  such that  $L = A \oplus B$ .

## 2 Preliminaries

All Lie algebras and vector spaces in this paper are considered over a fixed algebraically closed field  $\mathbb{K}$  of characteristic 0.

**Definition 2.1** *Let  $I$  be a set with a partial ordering  $\leq$ . Suppose  $I$  is directed upwards, i.e., for every  $i, j \in I$  there exists  $k \in I$  such that  $i \leq k$  and  $j \leq k$ . Let  $S = \{S_i : i \in I\}$  be a family of sets such that for every pair  $(i, j) \in I \times I$  with  $j \geq i$ , there is a map  $\pi_{ji} : S_j \rightarrow S_i$  satisfying the following two conditions:*

1.  $\pi_{ii}$  is the identity map for every  $i \in I$ ;
2. if  $i \leq j \leq k$ , then  $\pi_{ki} = \pi_{ji} \circ \pi_{kj}$ .

Let  $\pi = \{\pi_{ji}, i, j \in I, j \geq i : \pi_{ji} : S_j \rightarrow S_i\}$

Then  $(S, \pi)$  or  $(S_i, \pi_{ji})_{i, j \in I}$  or simply  $(S_i)$  is called an inverse system and the maps  $\pi_{ji} : S_j \rightarrow S_i$  are called the transition maps of the inverse system.

The inverse limit of this system, denoted by  $\varprojlim S_i$ , is the subset of the Cartesian product  $\prod_{i \in I} S_i$  consisting of all elements  $s = (s_i)_{i \in I}$  such that  $\pi_{ji}(s_j) = s_i$  for every  $j \geq i$ . In other words,  $\varprojlim S_i$  is the set of all families of elements  $(s_i)_{i \in I}$  which are compatible with the transition maps  $\pi_{ji} : S_j \rightarrow S_i$ .

**Remark 2.2** *If  $(S_i)$  is an inverse system, let  $\pi_i : \varprojlim S_i \rightarrow S_i$  be the canonical projection sending  $(s_i)$  to  $s_i$ . Then  $\pi_i = \pi_{ji} \circ \pi_j$  for every  $j \geq i$ .*

**Remark 2.3** *A surjective inverse system is an inverse system whose transition maps are surjective.*

**Remark 2.4** *An inverse sequence is an inverse system whose index set is  $I = \mathbb{N}$  (directed by its natural order).*

**Definition 2.5** *Inverse limit topology Let  $(S_i, \pi_{ji})$  be an inverse system where each set  $S_i$  is a topological space and each transition map is continuous. Then the inverse limit topology on  $\varprojlim S_i$  is the induced topology inherited from the product topology on  $\prod_{i \in I} S_i$  which has a basis consisting of the sets of the form  $\prod_{i \in I_i} U_i$ , with  $U_i$  is an open subset of  $S_i$  for every  $i \in I$  and  $U_i = S_i$  for all but a finitely many  $i \in I$ .*

**Remark 2.6** *The inverse limit topology is the smallest topology that makes each projection map  $\pi_i : \varprojlim S_i \rightarrow S_i$  continuous. A basis for the topology of  $S = \varprojlim S_i$  is given by the sets  $\pi_i^{-1}(U_i)$ , as  $i$  varies, where  $U_i$  is open in  $S_i$ .*

**Remark 2.7** Let  $A$  be a subset of  $\varprojlim S_i$  and let  $A_i$  be the image of  $A$  under the canonical projection  $\pi_i : \varprojlim S_i \rightarrow S_i$ . Then the closure of  $A$  is given by  $\overline{A} = \varprojlim \overline{A_i}$ . In particular, if  $A$  is closed in  $\varprojlim S_i$ , then  $A = \overline{A} = \varprojlim \overline{A_i} = \varprojlim A_i = \cap(A + \text{Ker } \pi_i)$ . Moreover, if  $A = \varprojlim C_i$  where  $(C_i)$  is a closed inverse subsystem of  $(S_i)$ , then  $A$  is closed in  $\varprojlim S_i$ . If each  $S_i$  is given the discrete topology, the resulting inverse limit topology on  $\varprojlim S_i$  is called the prodiscrete topology. Let  $(S_i, \pi_{ji})$  and  $(S_i, \pi'_{ji})$  be two inverse systems over the same index set  $I$ . If for every  $i \in I$ , the map  $f_i : S_i \rightarrow S'_i$  is continuous, then  $\varprojlim f_i : \varprojlim S_i \rightarrow \varprojlim S'_i$  is also continuous. Moreover, if each  $S_i$  is a topological group and each transition map is a continuous homomorphism, then  $\varprojlim S_i$  is a topological group.

**Definition 2.8** (The Coset Topology on Vector Spaces) Let  $V$  be a finite dimensional vector space over an arbitrary field  $F$ . There is a topology on  $V$  whose closed sets are the finite unions of sets of the form  $x + W$ , where  $x$  ranges over  $V$  and  $W$  ranges over the subspaces of  $V$ .

**Remark 2.9** The vector space  $V$  equipped with the coset topology is a compact  $T_1$  space. Also, every linear transformation from  $V$  into another such space is continuous and sends closed sets onto closed sets (since linear transformations over any field send subspaces to subspaces).

**Definition 2.10** (Hochschild-Mostow inverse system) An inverse system  $(X, \pi)$  of non-empty sets is called a super inverse system, or a Hochschild-Mostow inverse system, or a compact system if each  $X_i$  can be equipped with a compact  $T_1$  topology such that each transition map is a closed continuous map.

**Example 2.11** Every inverse system of finite dimensional vector spaces (with linear transition maps) is a super inverse system via the coset topology described before.

**Definition 2.12** A proper map  $f : X \rightarrow Y$  is a closed map such that each set  $f^{-1}(y)$ , where  $y \in Y$ , is compact in  $X$ .

**Proposition 2.13** If  $X$  is compact,  $Y$  is a  $T_1$  space, and  $f : X \rightarrow Y$  is a closed and continuous map, then  $f$  is proper.

*Proof.* Since  $Y$  is  $T_1$  space so every singleton subset of  $Y$  is closed,  $f$  is continuous, so  $f^{-1}(y)$  is also closed in the compact space  $X$ , thus  $f^{-1}(y)$  is compact in  $X$  and  $f$  is a proper map.

**Proposition 2.14** Let  $f : X \rightarrow Y$  be a continuous map of topological spaces. If  $X$  is compact,  $Y$  is a  $T_1$  space,  $\{A_i\}$  is a chain of closed subsets of  $X$ , then  $f(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f(A_i)$ .

*Proof.* It is obvious that  $f(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} f(A_i)$ . On the other hand, let  $y \in \bigcap_{i \in I} f(A_i)$  so  $y = f(x_i)$  with  $x_i \in A_i$  for all  $i \in I$ . Then  $y = f(x)$  where  $x \in A_i$  for all  $i \in I$ : Consider the collection  $\{f^{-1}(y) \cap A_i\}_{i \in I}$ . Since  $\{y\}$  is closed in the  $T_1$  space  $Y$  and  $f$  is continuous then  $f^{-1}(y)$  is closed in  $X$ . Hence  $f^{-1}(y) \cap A_i$  is closed in  $X$  for all  $i \in I$ . Also,  $x_i \in f^{-1}(y) \cap A_i$  for all  $i \in I$ . Since  $\{A_i\}$  is a chain, then  $\{f^{-1}(y) \cap A_i\}$  is a chain of non-empty closed subsets of  $X$ . Take a finite subset  $\{f^{-1}(y) \cap A_{ij}\}_{j=1,2,\dots,n}$  of  $\{f^{-1}(y) \cap A_i\}$ . Since we have a chain, then all elements are comparable, so there exists a smallest  $f^{-1}(y) \cap A_{il}$  for some  $l \in \{1, 2, \dots, n\}$ . So  $\bigcap_{j=1}^n (f^{-1}(y) \cap A_{ij}) = f^{-1}(y) \cap A_{il}$  is non-empty, so the finite intersection property holds for the collection  $\{f^{-1}(y) \cap A_i\}$ , but  $X$  is compact, then the intersection of the whole collection is non-empty, i.e. there exists  $x \in f^{-1}(y) \cap A_i$  for all  $i \in I$  such that  $y = f(x)$  and  $x \in \bigcap_{i \in I} A_i$ . Thus  $f(x) \in f(\bigcap_{i \in I} A_i)$  which implies that  $y \in f(\bigcap_{i \in I} A_i)$  and  $\bigcap_{i \in I} f(A_i) \subseteq f(\bigcap_{i \in I} A_i)$ .

**Theorem 2.15** (*The Projective Limit Theorem*) [4], p.501. Let  $(V_i, \pi_{ji})$  be a super inverse system consisting of non-empty compact  $T_1$  spaces and continuous closed maps  $\pi_{ji} : V_j \rightarrow V_i$ . Then

- (i)  $\varprojlim V_i \neq \emptyset$ .
- (ii) If each  $\pi_{ji}$  is surjective, then each canonical map projection  $\pi_i : \varprojlim V_i \rightarrow V_i$  is surjective.
- (iii) If  $v_i \in V_i$  is an essential element, then  $v_i$  is super essential.

**Theorem 2.16** [5], p.23. Let  $L$  be a finite dimensional semisimple Lie algebra over  $\mathbb{K}$ . Then

- (i) There exist simple ideals  $L_1, L_2, \dots, L_t$  [unique up to order] such that  $L = L_1 \oplus \dots \oplus L_t$ . Moreover, each ideal of  $L$  is a sum of such simple ideals.
- (ii) All ideals and homomorphic images of  $L$  are semisimple.

### 3 Closed Ideals of Prosemisimple Lie Algebras

**Theorem 3.1** Let  $(X_i, f_{ji})$  be a super inverse system. Then there exists a subinverse system of  $(X_i, f_{ji})$  with surjective transition maps whose inverse limit is equal to  $\varprojlim X_i$ .

*Proof.* Define  $Y_i = \bigcap_{j \succ i} f_{ji}(X_j)$  and  $g_{il} = f_{il/Y_i}$  (restriction of  $f_{il}$  to  $Y_i$ ). Then  $(Y, g)$  is a surjective inverse system:  $g_{il}$  are proper as  $f_{il}$  are, and by Proposition 2.14 we have:  $g_{il}(Y_i) = g_{il}(\bigcap_{j \succ i} f_{ji}(X_j)) = \bigcap_{j \succ i} g_{il}(f_{ji}(X_j)) = \bigcap_{j \succ i} f_{il}(f_{ji}(X_j)) = \bigcap_{i \succ l} f_{il}(X_i) = Y_l(i \succ j \succ l)$ . Now,  $Y_i$  is a closed subset of  $X_i$  since the maps  $f_{ji}$  are closed and arbitrary intersections of closed subsets are closed. Also if  $j \succ l \succ i$ , then  $f_{ji}(X_j) \subset f_{li}(X_l)$

so the sets  $f_{ji}(X_j)$  for  $j \succ i$  form a decreasing directed system of non-empty closed subsets of compact space  $X_i$ , hence their intersection  $Y_i$  is also non-empty and compact. Thus  $(Y, g)$  is a closed surjective subsystem of  $(X, f)$  in which  $Y_i$  are non-empty. Trivially  $Y = \varprojlim Y_i \subset \varprojlim X_i$ . Conversely, if  $(x) \in \varprojlim X_i$ , then  $x_j \in X_j$  for each  $j \in I$ . Since  $f_{jl}(x_j) = x_l$  whenever  $j \succ l$ , then  $x_l \in f_{jl}(X_j)$  for all  $j \succ l$ , thus  $x_l \in \bigcap_{j \succ l} f_{jl}(X_j) = Y_l$  for all  $l \in I$ , so  $g_{kj}(x_k) = f_{kj}(x_k) = x_j$  for all  $j \in I$  and  $k \succ j$ , hence  $(x) \in \varprojlim Y_i$  and  $\varprojlim Y_i = \varprojlim X_i$ .

**Theorem 3.2** *Let  $L$  be the inverse limit of a surjective inverse system of finite dimensional semisimple Lie algebras.  $A$  is a closed ideal of  $L$ . Then, there exists a unique ideal  $B$  of  $L$  such that  $L = A \oplus B$ .*

*Proof.* Since  $A$  is closed,  $A = \varprojlim A_i$ , where the  $A_i$  is the  $i$  th projection of  $A$ . Since  $L_i$  is semisimple by Theorem 2.16,  $L_i = \bigoplus_{t=1}^p S_t^i$ ,  $p \in \mathbb{N}$  where the  $S_t^i$  are simple ideals of  $L_i$ , and since  $A$  is an ideal of  $L$ ,  $A_i$  is an ideal of  $L_i$ . Thus by Theorem 2.16  $A_i = \bigoplus_{t=1}^{p'} S_t^i$ ,  $1 \leq p' \leq p$ . Similarly  $L_j = \bigoplus_{r=1}^m S_r^j$  and  $A_j = \bigoplus_{r=1}^{m'} S_r^j$ ,  $1 \leq m' \leq m$ . Let  $B_i = \bigoplus_{t=p'+1}^p S_t^i$ , then  $L_i = A_i \oplus B_i$ .

We note first that for a simple ideal  $S_t^i$  in  $A_i$ , there exists a unique  $S_r^j$  in  $A_j$  such that  $\pi_{j,i}(S_r^j) = S_t^i$ . This is due to the fact that the preimage of  $S_t^i$  is an ideal of  $L_j$ , thus it is a sum of some simple  $S_r^j$ , say for instance,  $m \geq 2$ , and  $\pi_{j,i}(S_1^j \oplus S_2^j) = S_t^i$ . Let  $\pi'_{j,i}$  be the restriction of  $\pi_{j,i}$  to  $S_1^j \oplus S_2^j$ . Then, Kernel  $\pi'_{j,i}$ , which is an ideal of  $S_1^j$ , is either 0, which is not the case since  $\pi'_{j,i}$  cannot be injective, or  $S_1^j \oplus S_2^j$ , which is also not the case since  $S_t^i \neq 0$ , thus the final choice is that Kernel  $\pi'_{j,i}$  is either  $S_1^j$  or  $S_2^j$ . Hence, there exists only one  $S_r^j$  such that  $\pi'_{j,i}(S_r^j) = S_t^i$ . Clearly this  $S_r^j$  must be in  $A_j$ . Similarly, for a simple ideal  $S_{t'}^i$  of  $B_i$ , there exists a unique simple ideal  $S_{r'}^j$  in  $B_j$  such that  $\pi_{j,i}(S_{r'}^j) = S_{t'}^i$ . Let  $\pi''_{j,i}$  be the restriction of  $\pi_{j,i}$  to  $B_j$ . If  $\pi''_{j,i}(B_j) = B_i$ , and  $\pi''_{k,j}(B_k) = B_j$ . Then,  $\pi''_{k,i}(B_k) = \pi_{k,i}(B_k) = (\pi_{j,i} \circ \pi_{k,j})B_k = \pi_{j,i}(B_j) = B_i$ . Thus,  $\pi''_{k,i} = \pi''_{j,i} \circ \pi''_{k,j}$ . Therefore, the  $B_i$  form a surjective subinverse system with transition maps  $\pi''_{j,i}$ , the restriction of  $\pi_{j,i}$  to  $B_j$ . Let  $B = \varprojlim B_i$ . We claim that  $L = A \oplus B$ . Let  $l \in L$ ,  $l = (l_i)$  and  $l_i$  can be written uniquely as  $a_i + b_i$ . Thus  $l = (a_i + b_i)$ . The  $a_i$  (respectively the  $b_i$ ) form a compatible inverse system. This is due to the fact that the  $l_i$  are compatible. Let  $\widehat{\pi}_{j,i}$ ,  $\widehat{\pi}'_{j,i}$  be the restrictions of  $\pi_{j,i}$  to  $a_i$  and  $b_i$  respectively. If  $\widehat{\pi}_{j,i}(a_j) = a_i$  and  $\widehat{\pi}_{k,j}(a_k) = a_j$ , and  $\widehat{\pi}'_{j,i}(b_j) = b_i$  and  $\widehat{\pi}'_{k,j}(b_k) = b_j$ , then, since  $\pi_{k,i}(l_k) = l_i$ , we get  $\pi_{k,i}(l_k) = l_i$  i.e.,  $\pi_{k,i}(a_k + b_k) = a_i + b_i$ , thus  $\pi_{k,i}(a_k) + \pi_{k,i}(b_k) = a_i + b_i$  i.e.  $\pi_{k,i}(a_k) - a_i = b_i - \pi_{k,i}(b_k)$ , but  $A_i \cap B_i = \{0\}$ . Thus  $\pi_{k,i}(a_k) = a_i$  and  $b_i = \pi_{k,i}(b_k)$ . Hence,  $\widehat{\pi}_{k,i}(a_k) = a_i$  and  $b_i = \widehat{\pi}'(b_k)$ . Therefore,  $l = (l_i) = (a_i + b_i) = (a_i) + (b_i)$ , and consequently  $L = A \oplus B$ .

Note that given any ideal  $M_i$  of  $L_i$ ,  $M_i = A_i \cap M_i \oplus B_i \cap M_i$ .

Suppose now that there exists an ideal  $C \neq 0$  of  $L$  such that  $L = A \oplus C$ . Let  $C_i$  be the  $i$  th projection of  $C$ . Then  $C_i = A_i \cap C_i \oplus B_i \cap C_i$ . But  $L_i = A_i \oplus C_i$ , thus  $L_i = A_i \oplus B_i \cap C_i = A_i \oplus B_i = A_i \oplus C_i$ , hence  $B_i \cap C_i = B_i = C_i$ , and consequently  $C = B$ .

**Corollary 3.3** *Let  $L$  be the inverse limit of an inverse system of finite dimensional semisimple Lie algebras.  $A$  is a closed ideal of  $L$ . Then, there exists a unique ideal  $B$  of  $L$  such that  $L = A \oplus B$ .*

*Proof.* This follows directly by combining Theorem 3.1 and Theorem 3.2.

**Corollary 3.4** *Let  $L = \varprojlim L_i$  be the inverse limit of an inverse sequence of finite dimensional semisimple Lie algebras. Suppose that every ideal of  $L$  is closed. Then every finite dimensional homomorphic image of  $L$  is semisimple.*

*Proof.* Let  $A$  be an ideal of  $L$ . Since  $A$  is closed, there exists a unique prosemisimple ideal  $B$  of  $L$  such that  $L = A \oplus B$ . Thus  $L/A$ , which is isomorphic to  $B$ , is prosemisimple, but  $L/A$  is finite dimensional, thus  $L/A$  is semisimple.

**Corollary 3.5** *Let  $L = \varprojlim L_i$ , be the inverse limit of an inverse sequence of finite dimensional semisimple Lie algebras. Let  $f: L \rightarrow M$  be a surjective Lie algebra homomorphism, where  $M$  is finite dimensional and  $\text{Ker } f$  is closed. Then  $M$  is semisimple.*

*Proof.* By the previous corollary  $L/\text{Ker } f$  is semisimple. But  $L/\text{Ker } f \cong M$ , thus  $M$  is semisimple.

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