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2-Simple Injective Rings

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Abstract

A ring R is called right 2-simple injective if, for every 2-generated right ideal I of R, every R-linear map from I to R with simple image extends to R. The class of right 2-simple injective rings is broader than that of right simple injective rings and right 2-injective rings . We study characterizations and properties of right 2-simple injective rings, several conditions under which right 2-simple injective rings are QF-rings are given.

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Throughout this article, R is an associative ring with identity, and all modules are unitary. As usual, $J(R)$ or J for short, $Z(_RR)(Z(R_R))$ and $Soc(_RR)(Soc(R_R))$ denote respectively the Jacobson radical, the left (right) singular ideal and the left (right) socle of R. The left annihilators of a subset X of R is denoted by $l_R(X)$ (or $l(X)$ for short), and the right annihilators of a subset X of R is denoted by $r_R(X)$ (or $r(X)$ for short). If M is an R-module, the notation $N \subseteq^{max} M$ means that N is a maximal submodule of M, and we write $N \subseteq^{\oplus} M$ if N is a direct summand of M for convenience.

Firstly, we recall some concepts. A ring R is called quasi-Frobenius, briefly QF, if it is right (or left) artinian (or noetherian), and right (or left) self-injective. A module M_R is called FP-injective (or absolutely pure) if, for any finitely generated submodule K of a free right R-module F, every R-homomorphism $K_R \to M_R$ extends to a homomorphism $F_R \to M_R$. A ring R is called right FP-injective if R_R is FP-injective. A ring R is called right simple injective if for every right ideal I of R, every R-linear map $\gamma : I \to R$ with $\gamma(I)$ simple extends to R. Let n be a positive integer, a ring R is called right n-injective if, for any n-generated right ideal I of R, every R-homomorphism from I to R extends to an R-homomorphism from R to R. Right 1-injective rings are called right P-injective [6]. A ring R is called right general principally injective (briefly right GP-injective) [5] if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R. GP-injective rings defined here are also called YJ-injective rings in [14]. A ring R is called right AGP-injective [11] if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and Ra^n is a direct summand of $l(r(a^n))$. A ring R is called right mininjective [7] if for any minimal right ideal I of R, every R-homomorphism from I to R extends to an R-homomorphism from R to R. The following implications hold:

right self-injective ⇒ right simple injective and right FP-injective.

right FP-injective ⇒ right 2-injective ⇒ right P-injective ⇒ right GP-injective⇒ right right AGP-injective and right mininjective.

Since right FP- injective rings need not be right self-injective [8, Example 5.45] and the integer ring $\mathbb{Z}$ is a right simple injective ring that is not right self-injective, right simple injective or right 2-injective rings need not be right self-injective. Right P-injective rings need not be right 2-injective [12, Example 1], right GP-injective rings need not be right P-injective [3], right AGP-injective rings need not be right GP-injective [11, Example 1.5], and right mininjective rings need not be right GP-injective neither (for example, the integer ring $\mathbb{Z}$ is right mininjective but it is not right GP-injective).

2-injective rings and simple injective rings and the relationships of them with QF-rings have been studied by many authors, see [1, 2, 6, 8, 9, 10, 12, 13]. In this article, we generalize the concepts of right 2-injective rings and right simple injective rings to right 2-simple injective rings, some characterizations and properties of them are studied, and several conditions under which 2-simple injective rings are QF-rings are given, many of them extend known results on right simple injective rings and right 2-injective rings.

We start with the following definition.

Definition 1. *A ring R is called right 2-simple injective if, for every 2-generated right ideal I of R, every R-linear map $\gamma : I \to R$ with $\gamma(I)$ simple extends to an endomorphism of R.*

Clearly, right simple injective rings and right 2-injective rings are both right 2-simple injective, but right 2-simple injective rings need neither be right simple injective nor right 2- injective. We recall that if R is a ring and ${}_RV_R$ is a bimodule, then the trivial extension $T(R,V)$ of R by V is the additive group $R \oplus V$ with multiplication given by $(r,v)(s,w) = (rs, rw + vs)$.

Example 2. *Let* $\mathbb{Z}_{2^\propto} = \{\frac{m}{2^i} + \mathbb{Z} \mid m \in \mathbb{Z}, i \in \mathbb{Z}^+\}$ *be the prüfer group of type* $2^\propto$*. Then by [9, Example 4], the trivial extension* $T(\mathbb{Z}, \mathbb{Z}_{2^\propto})$ *of* $\mathbb{Z}$ *by* $\mathbb{Z}_{2^\propto}$ *is a commutative simple-injective ring with simple essential socle, of course, it is 2-simple injective, but it is not* P*-injective.*

Recall that a ring R is called right *Kasch* if every simple right R-module embeds in R, equivalently if $l(T) \neq 0$ for every maximal right ideal T of R. Left Kasch rings can be defined similarly. R is call Kasch if it is left and right Kasch.

Example 3. *Let* $R = \mathbb{Z}_2[x_1, x_2, ...]$*, where the* x_i *are commuting indeterminants satisfying the relations* $x_i^3 = 0$ *for all* i*,* $x_i x_j = 0$ *for all* $i \neq j$*, and* $x_i^2 = x_j^2$ *for all* i *and* j*. Write* $m = x_1^2 = x_2^2 = \cdots$*. Then by [8, Example 2.6],* R *is a commutative local ring,* $J = span_{\mathbb{Z}_2}\{m, x_1, x_2, ...\}, J^3 = 0$*,* $Soc(R) = J^2 = Fm$ *is simple and essential in* R*. And by [8, Example 5.45],* R *is an* FP*-injective ring. So* R *is a commutative , Kasch, semiprimary 2-simple injective ring with simple essential socle, but it is not simple injective by [8, Example 6.2].*

It is easy to see that right 2-simple injective rings are right mininjective, but the inverse implications is not true. Indeed, even right P-injective rings need not be right 2-simple injective. Recall a ring R is called right *minfull* [7] if it is semiperfect, right mininjective and $Soc(eR) \neq 0$ for each local idempotent $e \in R$.

Example 4. *Let* K *be a field and* L *be a proper subfield of* K *such that* $\rho : K \to L$ *is an isomorphism, and let* $K[\rho; x]$ *be the ring of twisted left polynomials over* K *where* $xk = \rho(k)x$ *for all* $k \in K$*. Set* $R = K[\rho; x]/(x^2)$*. Then* R *is right* P*-injective, but* R *is not right 2-simple injective.*

Proof. By Rutter [12, Example 1], R is a right P-injective , left artinian local ring with only two idempotents 0 and 1 but R is not QF. Hence R is right minfull. By [7, Theorem 3.7(1)], R is right Kasch . If R is right 2-simple injective, then by Theorem 10(1), R is left P-injective , and thus R is left and right mininjective and left artinian. It follows that R is QF by [7, Corollary 4.8], a contradiction. □

Proposition 5. *A direct product* $\prod_{i\in I} R_i$ *of rings is right 2-simple injective if and only if* R_i *is right 2-simple injective for each* $i \in I$.

Proof. The proof of the necessity is trivial. Conversely, write $R = \prod_{i\in I} R_i$, assume that each R_i is right 2-simple injective, and let $\gamma : T \to R$ be a right R-homomorphism, where $T = Ra + Rb$ is a 2-generated right ideal of R and $\gamma(T)$ is simple. Let σ_i and π_i be the canonical inclusion and projection maps for R. Write $a = \langle a_i\rangle, b = \langle b_i\rangle$. For each $k \in I$, write $e_k = \sigma_i\pi_i(1)$ and define $T_k = \{x \in R_k \mid \sigma_k(x) \in T\}$. Then $T_k = R_k a_k + R_k b_k$ is a 2-generated right ideal of R_k, the map $\gamma_k = \pi_k\gamma\sigma_k : T_k \to R_k$ is a right R_k-homomorphism, and $\gamma_k(T_k) = \pi_k(\gamma\sigma_k(T_k)) \subseteq \pi_k(\gamma(T))$ is simple or 0 as a right ideal of R_k because π_k is a ring homomorphism. By hypothesis $\gamma_k = c_k\cdot$ for some $c_k \in R_k$. Now for any $t \in T$, write $t = \langle t_i\rangle, \gamma(t) = \langle s_i\rangle$, then each $t_k \in T_k$ because $\sigma_k(t_k) = te_k \in T$, and $\gamma\sigma_k(t_k) = \gamma(te_k) = \gamma(t)e_k = \sigma_k(s_k)$. Hence, $c_k t_k = \gamma_k(t_k) = \pi_k\gamma\sigma_k(t_k) = \pi_k\sigma_k(s_k) = s_k$. Let $c = \langle c_i\rangle \in R$, then $ct = \langle c_i t_i\rangle = \langle s_i\rangle = \gamma(t)$, so $\gamma = c\cdot$, as required. □

Example 6. *Let* R_1 *as in Example 2,* R_2 *as in Example 3, and let* $R = R_1 \times R_2$. *Then* R *is a commutative 2-simple injective ring which is neither simple injective nor 2-injective.*

Proof. Since R_1 and R_2 are both commutative 2-simple injective rings, by Proposition 5, R is a commutative 2-simple injective ring. With a similar way as the proof of Proposition 5, one can prove that a direct product of rings is 2-injective if and only if each of its factor is 2-injective. Note that a direct product of rings is simple-injective if and only if each of its factor is simple-injective by [8, Proposition 6.3], So if R is simple injective, then R_2 is simple injective, a contradiction. If R is 2-injective, then R_1 is 2-injective, a contradiction too. □

Proposition 7. *A ring* R *is right 2-simple injective if and only if an* R*-linear map* $\gamma : I \to R_R$ *extends to* $R_R \to R_R$ *whenever* I *is a 2-generated right ideal of* R *and* $\gamma(I)$ *is semisimple.*

Proof. Assume that R is right 2-simple injective. If $\gamma(I) = 0$ then $\gamma = 0$. Otherwise, let $\gamma(I) = K_1 \oplus \cdots \oplus K_n$, where the K_i are simple right ideals. If $\pi_i : \gamma(I) \to K_i$ is the projection, then $\pi_i\gamma = c_i\cdot$ for some $c_i \in R$ by hypothesis. It is routine to verify that $\gamma = (c_1 + \cdots + c_n)\cdot$, as required. □

We call a module M_R *2-simple quasi-injective* if , for any 2-generated submodule $X \subseteq M$, any R-homomorphism $X \to M$ with simple image extends to an endomorphism of M.

Lemma 8. *If M_R is a 2-simple quasi-injective module and $End(M_R)$ is local, then $Soc(M_R)$ is either 0 or simple and essential in M.*

Proof. Suppose $Soc(M_R) \neq 0$, and let $K \subseteq M$ be simple. If $0 \neq T \subseteq M$ is a cyclic submodule, it suffices to show that $K \subseteq T$. If not, define $\gamma : K \oplus T \to M$ by $\gamma(k+t) = k$. By hypothesis, let $\hat{\gamma} \in End(M)$ extend γ. Then $(1_M - \hat{\gamma})(k) = 0$ for each $k \in K$, so $\hat{\gamma} \notin J(End(M_R))$. Hence $\hat{\gamma}$ is invertible in the local ring $End(M_R)$, so the fact $\hat{\gamma}(T) = 0$ means $T = 0$, contrary to the assumption. □

If M_R is 2-simple quasi-injective and $N \subseteq^{\oplus} M$ then N_R is also 2-simple quasi-injective. Indeed, if X is a 2-generated submodule of N and $\gamma : X \to N$ has simple image then γ extends to $\hat{\gamma} : M \to M$, so $\pi\hat{\gamma}|_N : N \to N$ extends γ, where $\pi : M \to N$ is the projection.

Proposition 9. *Let R be a right 2-simple injective ring.*

(1) If e is a local idempotent in R then $Soc(eR)$ is either zero or simple and essential in eR.

(2) If R is semiperfect, then following conditions are equivalent:

(a) $Soc(eR) \neq 0$ for each local idempotent e.

(b) $Soc(R_R)$ is finitely generated and essential in R_R.

Proof. The module eR is 2-simple quasi-injective by the preceding remark, and $end(eR) \cong eRe$ is a local ring. hence (1) follows from Lemma 8. If $1 = e_1 + \cdots + e_n$, where the e_i are orthogonal local idempotents, then $Soc(R_R) = Soc(e_1R) \oplus \cdots \oplus Soc(e_nR)$ and (a) $\Rightarrow$ (b) follows from (1). The converse is clear. □

Theorem 10. *Let R be a right 2-simple injective, right Kasch ring. Then*

(1) R is left P-injective, and hence right and left mininjective.

(2) Ra is simple if and only if aR is simple. In particular, $Soc(R_R) = Soc({}_RR)$.

(3) $J(R) = Z({}_RR) = r_R(Soc(R_R))$.

(4) $l_R(J(R)) \trianglelefteq {}_RR$.

(5) If $e^2 = e$ is local then $Soc(Re)$ is simple and essential in Re.

(6) For the following conditions, we have (c) $\Rightarrow$ (a) $\Leftrightarrow$ (b):

(a) $Soc(eR) \neq 0$ for each local $e^2 = e \in R$.

(b) $Soc(eR)$ is simple and essential in eR for each local $e^2 = e \in R$.

(c) R is left Kasch.

(7) The map $\theta : T \mapsto l(T)$ gives a bijection from the set of maximal right ideals of R to the set of minimal left ideals of R, whose inverse map is given by $K \mapsto r(K)$.

Proof. (1). For every $a \in R$, we always have $aR \subseteq r_R l_R(a)$. If $b \in r_R l_R(a) - aR$, let $aR \subseteq T \subseteq^{max} (aR + bR)$. By the Kasch hypothesis, let $\sigma : (aR + bR)/T \to R$ be monic, and then define $\gamma : aR + bR \to R$ by $\gamma(x) = \sigma(x + T)$. Since $im(\gamma) = im(\sigma)$ is simple and R is right 2-simple injective, $\gamma = c\cdot$ for some $c \in R$. So $ca = \gamma(a) = 0$. This gives $cb = 0$ because $b \in r_R l_R(a)$. But $cb = \sigma(b + T) \neq 0$ because $b \notin T$, which is a contradiction. Hence $r_R l_R(a) = aR$, which shows that R is left P-injective by [6, Lemma 1.1].

(2). Since R is right 2-simple injective and right Kasch, R is right and left mininjective by (1), and then Ra is simple if and only if aR is simple by [7, Theorem 1.14(1)].

(3). Since R is left P-injective, $J(R) = Z({}_RR)$ by [6, Theorem 2.1]. For every maximal right ideal T of R. Since R is right Kasch, R/T can be embedded in R_R, thus for each $x \in r_R(Soc(R_R))$, $(R/T)x = 0$, and then $x \in J(R)$. Therefore, $J(R) = r_R(Soc(R_R))$.

(4). Let $0 \neq a \in R$. Suppose that T is a maximal submodule of aR. By the right Kasch hypothesis, let $\sigma : aR/T \to R$ be monic, and define $f : aR \to R$ by $f(x) = \sigma(x + T)$, then $im(f) = im(\sigma)$ is simple. Since R is right 2-simple injective, $f = c\cdot$ for some $c \in R$, and then $ca = f(a) = \sigma(a + T) \neq 0$. But $caJ(R) = f(aJ(R)) = f(0) = 0$, so $0 \neq ca \in Ra \cap l_R(J(R))$. And hence $l_R(J(R)) \trianglelefteq {}_RR$.

(5). First we have $l(J)e \cong Hom_R(eR/eJ, R)$ by [8, Lemma 3.1]. Since eR/eJ is simple (because e is local), and since R is right mininjective and right Kasch, by [8, Theorem 2.31], $l(J)e$ is a simple submodule of $Soc(Re)$. Hence (2) gives $l(J)e \subseteq Soc(Re) = Soc({}_RR) \cap Re = Soc({}_RR)e = Soc(R_R)e \subseteq l(J)e$. It follows that $Soc(Re) = l(J)e$ is simple and essential in Re by (4).

(6). Clearly $(b) \Rightarrow (a)$, and the converse follows from Proposition 9(1). To show that $(c) \Rightarrow (a)$, observe that $e.r(J) \cong Hom(Re/Je, R)$, so by [8, Theorem 2.31], $e.r(J)$ is simple because e is local and R is left mininjective by(1) and left Kasch by (c). This proves (a).

(7). Let $K = Rk$ be any minimal left ideal, then kR is a minimal right ideal by (2), and hence $lr(K) = K$ by [7, Lemma 1.1], therefore (7) follows by [7, Theorem 2.3]. □

We call a ring R left finite dimensional in case ${}_RR$ is of finite Goldie dimension.

Theorem 11. *Let R be a right 2-simple injective and right Kasch ring. Then R is left finite dimensional if and only if R is semilocal. In this case R has the following properties:*

(1) R is left Kasch.

(2) $Soc(R_R) = Soc(_RR)$ is finitely generated and essential both as a right and a left ideal. Hence R_R and $_RR$ are finitely cogenerated.

(3) $Soc(eR)$ is simple and essential in eR for each local $e^2 = e \in R$.

(4) $dim(_RR) = lenth[(R/J)_R]$.

Proof. If R is left finite dimensional, then by the right Kasch condition, R is semilocal by [8, Proposition 5.55]. Conversely, if R is semilocal, then by Theorem 10, $Soc(_RR) = Soc(R_R) = l(J) \trianglelefteq_R R$. Note that $Soc(R_R)$ is finitely generated as a left R-module by [8, Theorem 5.52] , so $Soc(_RR)$ is finitely generated as a left R-module, which follows that R is left finite dimensional.

(1). Since R is semilocal, right and left mininjective, and right Kasch, it is left Kasch by [8, Lemma 5.49].

(2). Since R is a semilocal, right and left mininjective ring, $Soc(_RR) = Soc(R_R) = r(J)$ is finitely generated as a left and as a right R-module by [8, Corollary 5.53]. And $Soc(_RR) \trianglelefteq_R R$ by our above discussion. But R is left Kasch by (1) and left P-injective by Theorem 10(1), so $Soc(_RR) = r(J) \trianglelefteq R_R$ by [6, Lemma 2.3].

(3). This follows from (1) and Theorem 10(6).

(4). Observe that $l(J) \cong Hom(R/J, R)$. But R/J is semisimple by hypothesis, say $R/J = K_1 \oplus \cdots \oplus K_n$, where each K_i is a simple right R-module. Hence $Soc(_RR) = Soc(R_R) = l(J) \cong Hom(R/J, R) = Hom(K_1 \oplus \cdots \oplus K_n, R) \cong Hom(K_1, R) \oplus \cdots \oplus Hom(K_n, R)$. Since R is right mininjective and right Kasch, by [8, Theorem 2.31(2)], each $Hom(K_i, R)$ is simple. Since $Soc(_RR) \trianglelefteq_R R$, this shows that $dim(_RR) = dim(_RSoc(_RR)) = n = length((R/J)_R)$. □

Corollary 12. *Let R be a right 2-simple injective and right Kasch ring. Then the following conditions are equivalent:*

(1) R is semilocal.

(2) R is left finitely cogenerated.

(3) R is left finite dimensional.

(4) R is right finitely cogenerated and left Kasch.

(5) R is right finite dimensional and left Kasch.

(6) R is right finite dimensional and right C_2.

(7) $Soc(R_R)$ is a finitely generated left ideal.

Proof. (1) $\Rightarrow$ (2), (4), (7), and (3) $\Rightarrow$ (1) by Theorem 11. (2) $\Rightarrow$ (3), and (4) $\Rightarrow$ (5) are clear. (5) $\Rightarrow$ (6) by [8, Proposition 1.46].

(6) $\Rightarrow$ (1) Since R is right finite dimensional, it contains no infinite set of orthogonal idempotents. By [8, Example 7.5], every monomorphism $R_R \to R_R$ is epic because R is right C_2. Then (1) follows from [8, Corollary C.3] as R is right finite dimensional.

(7) $\Rightarrow$ (1). Assume (7). Then by Theorem 10(2), $Soc(R_R) = Soc({}_RR)$ is a finitely generated left ideal. Let $Soc(R_R) = Ra_1 + Ra_2 + \cdots + Ra_n$, where Ra_i is a simple left ideal, $i = 1, 2, \cdots, n$. Then, by Theorem 10(3), $J = r_R(Soc(R_R)) = \cap_{i=1}^{n} r_R(a_i)$. Note that each $r_R(a_i) = r_R(Ra_i)$ is a maximal right ideal by Theorem 10(7), so R is semilocal. $\square$

Recall that a ring R is called a right *CS* ring if every right ideal of R is essential in a summand of R_R, a ring R is called a right *min-CS* ring if every minimal right ideal of R is essential in a summand of R_R, a ring R is called right C_2 if every right ideal of R that is isomorphic to a direct summand of R is itself a direct summand of R, a ring R is called right *continuous* if it is right *CS* and right C_2. We call a ring R a right *P-CS* ring if every principal right ideal of R is essential in a summand of R_R, We call a ring R right *P-continuous* if it is right *P-CS* and right C_2, and we call a ring R right *min-continuous* if it is right *min-CS* and right C_2.

Theorem 13. *Assume that R is a semiperfect, right 2-simple injective ring in which $Soc(eR) \neq 0$ for every local idempotent e of R. Then the following hold:*

(1) R is right and left Kasch.

(2) $Soc(R_R) = Soc({}_RR)$, which we denote as S.

(3) R is left P-injective.

(4) R is right P-continuous.

(5) $Soc(eR) = eS$ and $Soc(Re) = Se$ are simple and essential in eR and Re, respectively, for every local idempotent $e \in R$.

(6) If $e_1, ..., e_n$ are basic local idempotents then $\{e_1S, ..., e_nS\}$ and $\{Se_1, ..., Se_n\}$ are systems of distinct representatives of the simple right and left R-modules, respectively.

(7) R is right and left finitely cogenerated.

(8) $Z(R_R) = Z({}_RR) = J$.

Proof. As R is right minfull, it is right and left Kasch by [7, Theorem 3.7], proving (1) . Then (2) and (3) follows by Theorem 10. Moreover, $Soc({}_RR) \trianglelefteq R_R$ by Theorem 11(2), so (3) implies that R is a right *P-CS* ring by [8, Lemma 4.2] and [6, Lemma 1.1], this proves (4) because left Kasch ring is right C_2 by [8, Proposition 1.46]. Since R is right Kasch by (1), Theorem 10(5) shows that $Soc(Re) \neq 0$ for each local $e^2 = e \in R$, and then R is also left minfull because R is left mininjective by (3). And thus (5) follows from [8, Proposition

3.17(2)], Theorem 10(5) and Theorem 11(3). (6) follows from [8, Proposition 3.17(6)]. And (7) follows from Theorem 11(2). Finally, we have $Z(R_R) \subseteq J$ in any semiperfect ring because any right ideal not contained in J contains a nonzero idempotent, and $J \subseteq Z(R_R)$ because $S \trianglelefteq R_R$ by Theorem 11(2). Hence $Z(R_R) = J$; $Z(_RR) = J$ by (1) and Theorem 10(3). This proves (8). □

Lemma 14. *If R is a right* min-continuous *and right finitely cogenerated ring, then it is semiperfect and $Soc(R_R) \trianglelefteq R_R$.*

Proof. See the proof of (9) ⇒ (3) in [2, Theorem 2.3].

Corollary 15. *Let R be a right 2-simple injective ring. Then the following conditions are equivalent:*

(1) R is semiperfect and right Kasch.
(2) R is semiperfect and $Soc(R_R) \trianglelefteq R_R$.
(3) R is semiperfect and $Soc(R_R) \trianglelefteq_R R$.
(4) R is semiperfect and $Soc(eR) \neq 0$ for every local idempotent e of R.
(5) R is left min-CS and right Kasch.
(6) R is right P-continuous and right finitely cogenerated.
(7) R is right min-continuous and right finitely cogenerated.
(8) R is right P-continuous, left P-injective and left Kasch.
(9) R is right min-continuous, left P-injective and left Kasch.
(10) R is semilocal, right Kasch and right min-CS.

Proof. (1) ⇒ (2), (3) by Theorem 11(2). (2) ⇒ (4), (6) ⇒ (7), (8) ⇒ (9) , and (1), (7) ⇒ (10) are obvious . (4) ⇒ (1) by Theorem 13(1). (4) ⇒ (8) by Theorem 13. (7) ⇒ (2) by Lemma 14.

(3) ⇒ (1). Since $Soc(R_R) \trianglelefteq_R R$, $Soc(R_R) \cap Re \neq 0$ for every local idempotent $e \in R$. Let $0 \neq a \in Soc(R_R) \cap Re$, then $a = ae \in Soc(R_R)e$. Thus $Soc(R_R)e \neq 0$, and so R is right *Kasch* by [7, Proposition 3.3(2)].

(1), (3) ⇒ (5). Firstly, by (1), R is right Kasch. Secondly, since R is semiperfect and $Soc(R_R) \trianglelefteq_R R$, $lr(T)$ is essential in a summand of $_RR$ for each left ideal T of R. In particular, we have $lr(Ra)$ is essential in a summand of $_RR$ for each minimal left ideal Ra of R. But aR is a minimal left ideal by Theorem 10(2), and note that R is right minimal injective, so $lr(Ra) = Ra$ by [7, Lemma 1.1], which follows that R is left *min-CS*.

(5) ⇒ (1). Since R is right Kasch, right mininjective, by [8, Theorem 2.31], for every maximal right ideal M of R, $l(M)$ is a minimal left ideal, Which implies that $l(M)$ is essential in a summand of $_RR$ because R is left *min-CS*. Therefore, R is semiperfect by [8, Lemma 4.1].

$(1),(4) \Rightarrow (6)$. By Theorem 13(4), R is right P-continuous. By Theorem 11(2), R is right finitely cogenerated.

$(9) \Rightarrow (1)$. Let $L \subseteq^{max} {}_RR$. we show that $r(L)$ is essential in a summand of R_R. Since R is left $Kasch$, let $La = 0$, where $0 \neq a \in R$. Then $L = l(a)$, and thus $r(L) = rl(a) = aR$ as R is left P-injective. Since L is a maximal left ideal, Ra is a minimal left ideal, and hence aR is a minimal right ideal by [7, Theorem 1.14]. But R is right *min-CS*, aR is essential in a summand of R_R. This shows that $r(L)$ is essential in a summand of R_R. Hence R is semiperfect by [8, Lemma 4.1]. Moreover, R is right $Kasch$ by [8, Lemma 5.49].

$(10) \Rightarrow (9)$. Since R is right 2-simple injective and right Kasch, it is left P-injective by Theorem 10(1). Since R is also semilocal, it is left Kasch by [8, Lemma 5.49]. Note that left Kasch ring is right C_2 by [8, Proposition 1.46], so (9) follows. ☐

Lemma 16. *Let R be a left perfect right 2-simple injective ring. Then*

(1) R is right Kasch.

(2) R is left and right mininjective and $Soc(R_R) = Soc({}_RR)$ is essential both as a right and a left ideal.

Proof. (1). Since R is right 2-simple injective, it is right mininjective . Since R is left perfect, it is semiperfect and right semiartinian by [8, Theorem B.32], and then every nonzero right R-module has essential socle by [8, Theorem B.31]. In particular, $Soc(R_R) \trianglelefteq R_R$ and $Soc(eR) \neq 0$ for every local $e^2 = e \in R$. Therefore, R is right minfull, which implies that R is right Kasch by [7 ,Theorem 3.7].

(2). Since R is right 2-simple injective, right Kasch and semilocal, (2) follows from Theorem 10(1) and Theorem 11(2). ☐

A ring R is called right CF if every cyclic right R-module embeds in a free R-module. A ring R is called right coherent if every finitely generated right ideal of R is finitely presented. We call a ring R right *min-coherent* if every minimal right ideal of R is finitely presented. Next we give some applications of 2-simple injective rings to QF rings.

Theorem 17. *Let R be a right 2-simple injective ring, then the following statements are equivalent:*

(1) R is a QF-ring.

(2) R is right artinian.

(3) R is left artinian.

(4) R is left perfect and every cyclic right R-module is finite dimensional.

(5) R is left perfect, right min-coherent.

(6) R is a right noetherian ring with $Soc(R_R) \trianglelefteq R_R$.
(7) R has ACC on right annihilator and $Soc(R_R) \trianglelefteq R_R$.
(8) R is right Kasch with left annihilator ACC.
(9) R is right Kasch and left CF.
(10) R is left CS and left CF.
(11) R is semilocal and right CF.
(12) R is right GP-injective with right annihilator ACC.
(13) R is right AGP-injective with right annihilator ACC.
(14) R is left GP-injective with left annihilator ACC.
(15) R is left AGP-injective with left annihilator ACC.
(16) R is semiprimary with left annihilator ACC.
(17) R is semiprimary with right annihilator ACC.
(18) R is left and right perfect with left annihilator ACC.
(19) R is left perfect with left annihilator ACC.
(20) R is left perfect with right annihilator ACC.

Proof. (1) ⇒ (2) − (20) , (3) ⇒ (16) ⇒ (18) ⇒ (19), (6) ⇒ (7) , (17) ⇒ (20) , (12) ⇒ (13) , (14) ⇒ (15) are clear. (9) ⇒ (3) by [4, Corollary 2.6]. (10) ⇒ (3) by [4, Corollary 3.10]. (13) ⇒ (17) and (15) ⇒ (16) by [15, Corollary 1.6].

(5) ⇒ (2). Suppose (5) holds. Then R is left perfect and right mininjective and right min-coherent, so R is right Artinian by [10, Theorem 10].

(11) ⇒ (2). Since R is right 2-simple injective, it is right mininjective, hence $Soc(R_R) \subseteq Soc(_RR)$ by [7, Theorem 1.14(4)]. Therefore R is right artinian by [1, Theorem 2.10].

(2) ⇒ (4). Assume that R is right artinian. Then it is left perfect and each finitely generated right R-module is artinian, and so (4) follows.

(4) ⇒ (1). Since R is left perfect, by Lemma 16(2), it is left and right mininjective. Moreover, for any right ideal I of R, R/I is finite dimensional by hypothesis, so $Soc(R/I)$ is finite dimensional and then finitely cogenerated. But $Soc(R/I) \trianglelefteq R/I$ because R is right semiartinian, so R/I is finitely cogenerated. Therefore, R is right artinian and two-sided mininjective, and whence it is QF by [7, Corollary 4.8].

(19) ⇒ (1). Since R is left perfect, right 2-simple injective, by Lemma 16(2), it is left and right mininjective and $Soc(_RR) \trianglelefteq_R R$, and thus R is QF by [13, Theorem 2.5] because R has left annihilator ACC.

(8) ⇒ (1). Since R is right Kasch, it is left P-injective by Theorem 10(1). But R has ACC on left annihilator , it is right artinian by [8, Proposition 5.15] . Hence, R is left and right mininjective and right artinian . This follows that R is QF by [7, Corollary 4.8].

(7) ⇒ (17). Since right 2-simple injective ring is right mininjective and hence right minsymmetric, (17) follows by [13, Lemma 2.3].

(20) $\Rightarrow$ (1). By Lemma 16(2), R is left and right mininjective and $Soc(R_R) \trianglelefteq R_R$, and thus R is QF by [13, Theorem 2.5] because R has right annihilator ACC. □

Corollary 18 [12, Corollary 3]. *A ring R is QF if and only if R is right 2-injective with the ascending chain condition on annihilator right ideals.*

Proof. The sufficiency by Theorem 17(12), and the necessity is clear. □

Corollary 19 [13, Theorem 2.8]. *If R is a right simple injective ring with ACC on right annihilators in which* $Soc(R_R) \trianglelefteq R_R$*, then R is QF.*

Proof. By Theorem 17(7). □

Our following corollary extends some results of [1, Corollary 2.15].

Corollary 20. *The following conditions are equivalent for a left CF ring R:*
(1) R is QF.
(2) R is right Kasch and right 2-simple injective.
(3) R is left CS and right 2-simple injective.

Proof. By Theorem 17(9), (10). □

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