On $\alpha$-Rigid-Like Properties

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Abstract

This paper deals with well-known extensions of the $\alpha$-rigid-like properties to arbitrary rings. We investigate the transfer of these notions to some ring extensions and then generate original families of rings subject to various $\alpha$-rigid-like properties.

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1 Introduction

Throughout this paper, all rings are associative with identity element, and all modules are unital. We denote by $nil(R)$ the set of all nilpotent elements of $R$.

According to Krempa [6], an endomorphism $\alpha$ of a ring $R$ is called to be rigid, if $a \alpha (a) = 0$ implies $a = 0$ for $a \in R$. We call a ring $R$ $\alpha$-rigid, if there exists a rigid endomorphism $\alpha$ of $R$. Note that any rigid endomorphism of a ring is a monomorphism. Clearly every subring of a weak $\alpha$-rigid ring is a weak $\alpha$-rigid ring. See for instance [[6], [4]].

Motivated by results in [[6], [4]], Ouyang [7] introduced the weak $\alpha$-rigid rings which are a generalization of $\alpha$-rigid rings as follows: let $\alpha$ be an endomorphism of a ring $R$, $R$ is said to be weak $\alpha$-rigid, if:

$$a \alpha (a) \in nil(R) \iff a \in nil(R).$$

Note that a ring $R$ is $\alpha$-rigid if and only if it is weak $\alpha$-rigid and reduced (see [7, Proposition 2.2]).
Given a ring $A$ and an $A$-module $E$, the set $R := A \propto E$ of pairs $(a, e)$ with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$ is called the trivial ring extension of $A$ by $E$ (also called the idealization of $E$ over $A$). See for instance [1, 5].

Introduced by D’Anna and Fontana [2, 3], the amalgamated duplication of $R$ along an ideal $I$ is defined as the following subring with unit element $(1, 1)$ of $R \times R$:

$$R \bowtie I = \{(r, r + i) | r \in R, i \in I\}.$$ Note that if $I^2 = 0$, the two extensions rings $R \times I$ and $R \bowtie I$ coincide.

In this paper, we investigate the transfer of the $\propto$-rigid-like properties to trivial ring extensions and amalgamated duplication of a ring along an ideal and then generate original families of rings subject to various $\propto$-rigid-like properties.

2 Main Results

We begin this section with a result which studies the transfer of the $\propto$-rigid-like properties to trivial ring extension.

**Theorem 2.1** I) Let $A \subseteq B$ be an extension of rings, $\propto$ be an endomorphism of $B$ such that $\propto (A) \subseteq A$, $R := A \propto B$, and set $\bar{\propto}$ the endomorphism of the ring $R$ defined by setting $\bar{\propto} (x, y) = (\propto (x), \propto (y))$ for each $(x, y) \in R$. Then:

1. $R$ is a weak $\bar{\propto}$-rigid ring if and only if so is $A$.
2. $R$ is never an $\bar{\propto}$-rigid ring.

II) Let $A$ be a ring, $\propto$ an endomorphism of $A$, $I$ an ideal of $A$, $R := A \propto I$, and set $\bar{\propto}$ the endomorphism of the ring $R$ defined by setting $\bar{\propto} (x, y) = (\propto (x), \propto (y))$ for each $(x, y) \in R$. Then:

1. $R$ is a weak $\bar{\propto}$-rigid ring if and only if $A$ is a weak $\propto$-rigid ring.
2. $R$ is never an $\bar{\propto}$-rigid ring.

**Proof** I) 1) Since $A$ is identified to a subring of $R$, it is clear that $A$ is weak $\propto$-rigid provided $R$ is weak $\bar{\propto}$-rigid. Conversely, assume that $A$ is weak $\propto$-rigid. First, let $(x, y) \in R$ such that
\[(x, y)\tilde{\alpha}(x, y) \in \text{nil}(R).\] We have to show that \((x, y) \in \text{nil}(R).\) We have
\[(x, y)\tilde{\alpha}(x, y) = (x, y)(\alpha (x), \alpha (y)) = (x \alpha (x), x \alpha (y) + \alpha (x)y) \in \text{nil}(R).\] Then, \(x \alpha (x) \in \text{nil}(A).\) Therefore, \(x \in \text{nil}(A)\) since \(A\) is weak \(\alpha\)-rigid.

Consequently, \((x, y) \in \text{nil}(R),\) as desired.

Now, let \((x, y) \in \text{nil}(R).\) We have to prove that \((x, y)\tilde{\alpha}(x, y) \in \text{nil}(R).\) Since
\[(x, y) \in \text{nil}(R),\] then \(x \in \text{nil}(A).\) Hence, \(x \alpha (x) \in \text{nil}(A).\) Thus, since
\[(x, y)\tilde{\alpha}(x, y) = (x \alpha (x), x \alpha (y) + \alpha (x)y),\] it is clear that \((x, y)\tilde{\alpha}(x, y) \in \text{nil}(R),\) as desired.

2) Clearly, \(R\) is never a reduced ring since \(0 \neq (0 \alpha B) \subseteq \text{nil}(R).\) Hence, \(R\) is never an \(\tilde{\alpha}\)-rigid ring by [7, Proposition 2.2].

II) We mimic the proof of I) and this completes the proof of Theorem 2.1.

The following Corollary is an immediate consequence of Theorem 2.1 (I).

**Corollary 2.2** [7, Corollary 3.2]

Let \(A\) be a ring, \(\alpha\) an endomorphism of \(A, R := A \alpha A,\) and set \(\tilde{\alpha}\) the endomorphism of the ring \(R\) defined by setting \(\tilde{\alpha}(x, y) = (\alpha (x), \alpha (y))\) for each \((x, y) \in R.\) Then \(R\) is a weak \(\alpha\)-rigid ring if and only if \(A\) is a weak \(\alpha\)-rigid ring.

Also, Theorem 2.1 enriches the literature with new examples of non-\(\tilde{\alpha}\)-rigid weak \(\tilde{\alpha}\)-rigid rings, as shown below.

**Example 2.3** Let \(A\) be an \(\alpha\)-rigid ring and consider the same notations of Theorem 2.1. Then, the ring \(R\) is a weak \(\tilde{\alpha}\)-rigid ring which is not \(\tilde{\alpha}\)-rigid.

Now, we study the transfer of the \(\alpha\)-rigid-like properties in amalgamated duplication of a ring along an ideal.

**Theorem 2.4** Let \(A\) be a ring, \(\alpha\) an endomorphism of \(A, I\) an ideal of \(A, R := A \alpha I,\) and set \(\tilde{\alpha}\) the endomorphism of the ring \(R\) defined by setting \(\tilde{\alpha}(x, y) = (\alpha (x), \alpha (y))\) for each \((x, y) \in R.\) Then:

1. \(R\) is a weak \(\tilde{\alpha}\)-rigid ring if and only if so is \(A.\)

2. \(R\) is a \(\alpha\)-rigid ring if and only if so is \(A.\)

We need the following Lemma before proving Theorem 2.4.

**Lemma 2.5** Let \((R_i)_{i=1,2,...,n}\) be a family of rings and let \(R := \prod_{i=1}^n R_i.\) Then:
1. $R$ is a weak $\times$-rigid ring if and only if $R_i$ is weak $\propto_i$-rigid for every $i = 1, \ldots, n$ where $\times(a_1, a_2, \ldots, a_n) = (\propto (a_1), \propto (a_2), \ldots, \propto (a_n))$.

2. $R$ is an $\times$-rigid ring if and only if $R_i$ is $\propto_i$-rigid for every $i = 1, \ldots, n$, where $\times(a_1, a_2, \ldots, a_n) = (\propto (a_1), \propto (a_2), \ldots, \propto (a_n))$.

**Proof** We prove the result for $i = 1, 2$, and the Lemma 2.5 will be established by induction on $n$.

1. Assume that $R_1 \times R_2$ is a weak $\times$-rigid ring. Then $R_i$ is a weak $\propto_i$-rigid ring (since $R_i$ is a subring of a $(R_1 \times R_2)$ for $i = 1, 2$). Conversely, assume that $R_i$ is weak $\propto_i$-rigid for $i = 1, 2$. We wish to show that $R_1 \times R_2$ is a weak $\times$-rigid ring.

Let $(a_1, a_2) \times (a_1, a_2) = 0$. Then $(a_1 \propto_1 (a_1), a_2 \propto_2 (a_2)) = 0$ and so $a_1 \in \text{nil}(R_1)$ and $a_2 \in \text{nil}(R_2)$ (since $R_i$ is weak $\propto_i$-rigid for $i = 1, 2$). Hence, $(a_1, a_2) \in \text{nil}(R)$ since $\text{nil}(R) = \text{nil}(R_1) \times \text{nil}(R_2)$.

Now, let $(a_1, a_2) \in \text{nil}(R_1 \times R_2)$. Then $a_1 \in \text{nil}(R_1)$ and $a_2 \in \text{nil}(R_2)$ and so $a_1 \propto_1 (a_1) = 0$ and $a_2 \propto_2 (a_2) = 0$ (since $R_i$ is weak $\propto_i$-rigid for $i = 1, 2$). Hence, $(a_1, a_2) \times (a_1, a_2) = 0$, as desired.

2. Assume that $R_1 \times R_2$ is an $\times$-rigid ring, then $R_i$ is an $\propto_i$-rigid (since $R_i$ is a subring of $R_1 \times R_2$) for $i = 1, 2$.

Conversely, assume that $R_i$ is an $\propto_i$-rigid ring for $i = 1, 2$. We wish to show that $R_1 \times R_2$ is an $\times$-rigid ring. Let $0 = (a_1, a_2) \times (a_1, a_2) = (a_1, a_2)(\propto_1 (a_1), \propto_2 (a_2)) = (a_1 \propto_1 (a_1), a_2 \propto_2 (a_2))$. Then $a_1 = 0$ and $a_2 = 0$ (since $R_i$ is an $\propto_i$-rigid for $i = 1, 2$) and so $(a_1, a_2) = 0$, as desired.

**Proof of Theorem 2.4** 1) It is clear that $A$ is a subring of $A \propto I$, and $A \propto I$ is a subring of $A \times A$. On the other hand, $A$ is a weak $\propto$-rigid ring if and only if $A \times A$ is a weak $\times$-rigid ring by Lemma 2.5, where $\times$ is the natural extension of $\propto$ to the ring $A \times A$. Hence, it is clear that $A$ is a weak $\propto$-rigid ring if and only if $A \propto I$ is a weak $\times$-rigid ring (if and only if $A \times A$ is a weak $\times$-rigid ring).

2) We mimic the proof of 1) and this completes the proof of Theorem 2.4.

**Corollary 2.6** Let $A$ be a reduced ring, $I$ an ideal of $A$, and let $R := A \propto I$. With the same notations of Theorem 2.4, the following assertions are equivalents:

1) $A$ is an $\propto$-rigid ring.

2) $A$ is a weak $\propto$-rigid ring.

2) $R := A \propto I$ is an $\times$-rigid ring.

2) $R := A \propto I$ is a weak $\times$-rigid ring.
Proof Follows by using Theorem 2.4, [7, Proposition 2.2], and [2, Theorem 3.5 (a)(vi)].

Theorem 2.4 generates new and original examples of rings subject to various $\alpha$-rigid-like properties.

Example 2.7 Let $A$ be an non-reduced weak $\alpha$-rigid ring and $I$ be an ideal of $A$. Then $R := A \triangleright I$ is a weak $\alpha$-rigid ring which is not $\alpha$-rigid.

References


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