(α, β)-Cut of Intuitionistic Fuzzy Ideals

D. K. Basnet
Dept. of Mathematics, Assam University
Silchar-788011, Assam, India
dkbasnet@rediffmail.com

Abstract

For any Intuitionistic fuzzy set $A = \{ < x, \mu_A(x), \nu_A(x) > | x \in E \}$ of a set $E$, a $(\alpha, \beta)$-cut of $A$ is the crisp subset $\{ x \in E | \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta \}$ of $E$. In this paper some interesting properties of $(\alpha, \beta)$-cut of Intuitionistic fuzzy ideals of a ring were discussed.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy ideal, (α,β)-cut

1. Introduction

The idea of Intuitionistic fuzzy sets was initiated by K.T.Atanassov [1] and it was extended to Intuitionistic fuzzy ideals by Banerjee and Basnet [2, 3]. In theory of fuzzy sets the level subsets are very very important tools for development of the theory. The motivation of the paper is the same as that of level subsets in fuzzy set theory i.e. to establish some important links between Intuitionistic fuzzy sets and crisp sets. That might be a help for characterizing Intuitionistic fuzzy algebraic structures.

2. Preliminaries

Definition 2.1. [1] Let $E$ be a fixed nonempty set. An Intuitionistic Fuzzy Set (IFS) $A$ of $E$ is an object of the following form $A = \{ < x, \mu_A(x), \nu_A(x) > | x \in E \}$ where $\mu_A : E \rightarrow [0, 1]$ and $\nu_A : E \rightarrow [0, 1]$ define the degree of membership and degree of nonmembership of the element $x \in E$ respectively and for every $x \in E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.2. [1] If $A = \{ < x, \mu_A(x), \nu_A(x) > | x \in E \}$ and $B = \{ < x, \mu_B(x), \nu_B(x) > | x \in E \}$ be any two IFS of a set $E$ then , $A \subseteq B$ if and only if for all $x \in E$, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, $A = B$ if and only if for all $x \in E$, $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$,
A ∩ B = \{ x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x) > | x \in E \},
where (\mu_A \cap \mu_B)(x) = \min \{ \mu_A(x), \mu_B(x) \} and (\nu_A \cup \nu_B)(x) = \max \{ \nu_A(x), \nu_B(x) \},
A \cup B = \{ x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x) > | x \in E \},
where (\mu_A \cup \mu_B)(x) = \max \{ \mu_A(x), \mu_B(x) \} and (\nu_A \cap \nu_B)(x) = \min \{ \nu_A(x), \nu_B(x) \}.

Also we see that a fuzzy set has the form \{ x, \mu^c_A(x) > | x \in E \}, where \mu^c_A(x) = 1 - \mu_A(x).

Definition 2.3. [2] An IFS A = \{ x, \mu_A(x), \nu_A(x) > | x \in R \} of a ring R is said to be an Intuitionistic Fuzzy Ideal (in short IFI) of R if for all x, y \in R,
(i) \mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \}
(ii) \nu_A(xy) \geq \max \{ \nu_A(x), \nu_A(y) \}
(iii) \nu_A(x - y) \leq \max \{ \nu_A(x), \nu_A(y) \}
(iv) \mu_A(xy) \leq \min \{ \nu_A(x), \nu_A(y) \}

Theorem 2.4. [2] If A = \{ x, \mu_A(x), \nu_A(x) > | x \in R \} is an IFI of R then
\mu_A(0) \geq \mu_A(x), \nu_A(0) \leq \nu_A(x), \mu_A(-x) = \mu_A(x), \nu_A(-x) = \nu_A(x) for all x \in R.

Definition 2.5. [2] Let A = \{ x, \mu_A(x), \nu_A(x) > | x \in R \} and
B = \{ x, \mu_B(x), \nu_B(x) > | x \in R \} be two IFI’s of a ring R then their sum A + B is defined as A + B = \{ x, (\mu_A + \mu_B)(x), (\nu_A + \nu_B)(x) > | x \in R \} where
(\mu_A + \mu_B)(x) = \sup \{ \min \{ \mu_A(a), \mu_B(b) \} \} and (\nu_A + \nu_B)(x) = \inf \{ \max \{ \nu_A(a), \nu_B(b) \} \}

Definition 2.6. [2] Let A = \{ x, \mu_A(x), \nu_A(x) > | x \in R \} and
B = \{ x, \mu_B(x), \nu_B(x) > | x \in R \} be two IFI’s of a ring R then their product AB is defined as AB = \{ x, (\mu_A \mu_B)(x), (\nu_A \nu_B)(x) > | x \in R \} where,
(\mu_A \mu_B)(x) = \sup \{ \min \{ \mu_A(a), \mu_B(b) \} \} and
(\nu_A \nu_B)(x) = \inf \{ \max \{ \nu_A(a), \nu_B(b) \} \}

Theorem 2.7. [2] If A and B are two IFI’s of a ring R then A + B and AB are also IFI’s of R.

Definition 2.8. Let R and R’ be two rings and f : R → R’ be a homomorphism, and A = \{ x, \mu_A(x), \nu_A(x) > | x \in R \} and A’ = \{ y, \mu_A'(y), \nu_A'(y) > | y \in R’ \} be fuzzy subsets of R and R’ respectively, then the image f(\mu) and the inverse image f^{-1}(\mu) are defined as follows:
f(A) = \{ y, f(\mu_A(y)), f(\nu_A(y)) > | y \in R’ \} and f^{-1}(A) = \{ x, f^{-1}(\mu_A(x)), f^{-1}(\nu_A(x)) > | x \in R \},
where (f(\mu_A))(y) = \sup \{ \mu_A(x) : x \in f^{-1}(y) \} if f^{-1}(y) \neq \emptyset
= 0 if f^{-1}(y) = \emptyset
(f(\nu_A))(y) = \inf \{ \nu_A(x) : x \in f^{-1}(y) \} if f^{-1}(y) \neq \emptyset
= 1 if f^{-1}(y) = \emptyset
And f^{-1}(\mu_A)(x) = \mu_A(f(x)), f^{-1}(\nu_A)(x) = \nu_A(f(x))

3. (\alpha, \beta)-cut of an IFS.

Definition 3.1. For any Intuitionistic fuzzy set A = \{ x, \mu_A(x), \nu_A(x) > | x \in E \} of
Theorem 3.8. Thus x

Corollary 3.4. μ

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following example: Consider the ring (A = {x, y | x, y ∈ R}).

Proof. Let μA(0) ≥ α, νA(y) ≤ β. Clearly Cα, β(A) ≠ φ. Let x, y ∈ Cα, β(A). Then μA(x), μA(y) ≥ α and νA(x), νA(y) ≤ β. Now μA(x – y) ≥ min {μA(x), μA(y)} ≥ α and νA(x) ≤ max {νA(x), νA(y)} ≤ β.

Thus x – y ∈ Cα, β(A). Also if r ∈ R then μA(rx) ≥ max {μA(r), μA(x)} ≥ μA(x) ≥ α, (μA(x), μA(y)) ≥ μA(x) ≥ α and νA(rx) ≤ min {νA(r), νA(x)} ≤ νA(x) ≤ β.

Therefore rx, xr ∈ Cα, β(A). Hence Cα, β(A) is an ideal of R.

Theorem 3.3. If A is an IFI of R then Cα, β(A) ⊆ Cγ, δ(A) if α ≥ γ and β ≤ δ.

Proof. Let x ∈ Cα, β(A), then μA(x) ≥ α and νA(x) ≤ β. Since α ≥ γ and β ≤ δ, so μA(x) ≥ α ≥ γ and νA(x) ≤ β ≤ δ. Therefore x ∈ Cγ, δ(A). Hence Cα, β(A) ⊆ Cγ, δ(A).

Corollary 3.4. If α + β ≤ 1 then C1−β, β(A) ⊆ Cα, β(A) ⊆ Cα, 1−α(A).

Theorem 3.5. If Cα, β(A) is an ideal of R for all α, β ∈ [0, 1] with α + β ≤ 1 then A = {x < r, μA(x), νA(x) ≥ δ | x ∈ R} is an IFI of R.

Proof. Let x, y ∈ R and α = min {μA(x), μA(y)} and β = max {νA(x), νA(y)}.

Then μA(x) ≥ α, νA(x) ≤ β and μA(y) ≥ α, νA(y) ≤ β ⇒ x, y ∈ Cα, β(A). Since Cα, β(A) is an ideal of R so x – y ∈ Cα, β(A). Consequently μA(x – y) ≥ α = min {μA(x), μA(y)} and νA(x – y) ≤ β = max {νA(x), νA(y)}. Next let δ = max {μA(x), μA(y)} say δ = μA(x). Since μA(x) + νA(x) ≤ 1, so νA(x) ≤ 1 – μA(x) = 1 – δ.

Thus x ∈ Cδ, 1−δ (A). Since Cδ, 1−δ (A) is an ideal of R so xy ∈ Cδ, 1−δ (A). Hence μA(xy) ≥ δ = max {μA(x), μA(y)}. Similarly we can show νA(xy) ≤ min {νA(x), νA(y)}. Hence A is an IFI of R.

Theorem 3.6. If A and B are IFI’s of a ring R then Cα, β(A ∩ B) = Cα, β(A) ∩ Cα, β(B).

Proof. We have Cα, β(A ∩ B) = {x ∈ R | μA ∩ μB) (x) ≥ α, (νA ∩ νB) (x) ≤ β}. Now x ∈ Cα, β(A ∩ B) ⇔ (μA ∩ μB) (x) ≥ α and (νA ∩ νB) (x) ≤ β ⇔ min {μA(x), μB(y)} ≥ α and max {νA(x), νB(y)} ≤ β.

⇔ μA(x), μB(y) ≥ α and νA(x), νB(y) ≤ β ⇔ x ∈ Cα, β(A) and x ∈ Cα, β(B)

⇔ x ∈ Cα, β(A) ∩ Cα, β(B). Therefore Cα, β(A ∩ B) = Cα, β(A) ∩ Cα, β(B).

Theorem 3.7. If A ⊆ B then Cα, β(A) ⊆ Cα, β(B), where A and B are IFI’s of R.

Theorem 3.8. If A and B are IFI’s of R then Cα, β(A ∪ B) ⊇ Cα, β(A) ∪ Cα, β(B).

Proof. Since A ⊆ A ∪ B and B ⊆ A ∪ B so by theorem 3.7, Cα, β(A) ⊆ Cα, β(A ∪ B) and Cα, β(B) ⊆ Cα, β(A ∪ B) and therefore Cα, β(A ∪ B) ⊇ Cα, β(A) ∪ Cα, β(B).

Remark 3.9. However the reverse inclusion does not hold, as shown by the following example: Consider the ring (Z4, +, ×) where Z4 = {0, 1, 2, 3} and define μA(0) = 0.8, μA(2) = 0.5, μA(1) = μA(3) = 0.2; νA(0) = 0.1, νA(2) = 0.4, νA(1) = νA(3) = 0.6 and μB(0) = 0.9, μB(2) = 0.4, μB(1) = μB(3) = 0.3; νB(0) = 0.1, νB(2) = 0.2, νB(1) = νB(3) = 0.6.
Then \( A = \{x, \mu_A(x), v_A(x) > | x \in \mathbb{Z}_4\} \) and \( B = \{x, \mu_B(x), v_B(x) > | x \in \mathbb{Z}_4\} \) are IFI’s of \( \mathbb{Z}_4 \). Now \( C_{3, 2}(A) = \{0\} \) and \( C_{3, 2}(B) = \{0\} \).

Also \((\mu_A \cup \mu_B)(0) = 0.9, (\mu_A \cup \mu_B)(2) = 0.5, (\mu_A \cup \mu_B)(1) = (\mu_A \cup \mu_B)(3) = 0.3\) and \((v_A \vee v_B)(0) = 0.1, (v_A \vee v_B)(2) = 0.2, (v_A \vee v_B)(1) = (v_A \vee v_B)(3) = 0.6\). Since \( C_{3, 2}(A \cup B) = \{0, 2\} \) so \( C_{3, 2}(A) \cup C_{3, 2}(B) \neq C_{3, 2}(A \cup B) \).

**Remark 3.10.** The equality holds if \( \alpha + \beta = 1 \) as shown below:

Let \( x \in C_{\alpha, \beta}(A \cup B) \), then \((\mu_A \cup \mu_B)(x) \geq \alpha \Rightarrow \max\{\mu_A(x), \mu_B(x)\} \geq \alpha \Rightarrow \mu_A(x) \geq \alpha \text{ or } \mu_B(x) \geq \alpha \). If \( \mu_A(x) \geq \alpha \) then \( v_A(x) \leq 1 - \mu_A(x) \leq 1 - \alpha = \beta \) and so \( x \in C_{\alpha, \beta}(A) \subseteq C_{\alpha, \beta}(A) \cup C_{\alpha, \beta}(B) \). Similarly if \( \mu_B(x) \geq \alpha \) then \( x \in C_{\alpha, \beta}(B) \subseteq C_{\alpha, \beta}(A) \cup C_{\alpha, \beta}(B) \). Hence the equality follows.

**Theorem 3.11.** \( C_{\alpha, \beta}(\bigcap \{A_i | i \in I\}) = \bigcap \{C_{\alpha, \beta}(A_i) | i \in I\} \).

**Theorem 3.12.** For any two IFI’s \( A \) and \( B \) of a ring \( R \), \( C_{\alpha, \beta}(A) + C_{\alpha, \beta}(B) \subseteq C_{\alpha, \beta}(A + B) \) and the equality holds if \( \alpha + \beta = 1 \).

**Proof.** Let \( x = y + z \in C_{\alpha, \beta}(A) + C_{\alpha, \beta}(B) \), where \( y \in C_{\alpha, \beta}(A) \) and \( z \in C_{\alpha, \beta}(B) \). Then \( \mu_A(y) \geq \alpha, v_A(y) \leq \beta \), and \( \mu_B(z) \geq \alpha, v_B(z) \leq \beta \).

\[
\Rightarrow \min\{\mu_A(y), \mu_B(z)\} \geq \alpha \text{ and } \max\{v_A(y), v_B(z)\} \leq \beta.
\]

\[
\Rightarrow \sup\{\min\{\mu_A(y), \mu_B(z)\}\} \geq \alpha \text{ and } \inf\{\max\{v_A(y), v_B(z)\}\} \leq \beta.
\]

\[
\Rightarrow (\mu_A + \mu_B)(x) \geq \alpha \text{ and } (v_A + v_B)(x) \leq \beta.
\]

\[
\Rightarrow x \in C_{\alpha, \beta}(A) + C_{\alpha, \beta}(B).
\]

Hence \( C_{\alpha, \beta}(A) + C_{\alpha, \beta}(B) \subseteq C_{\alpha, \beta}(A + B) \).

For the other part let \( \alpha + \beta = 1 \) and \( x \in C_{\alpha, \beta}(A + B) \).

Then \((\mu_A + \mu_B)(x) \geq \alpha \text{ and } (v_A + v_B)(x) \leq \beta\).

Now \((\mu_A + \mu_B)(x) \geq \alpha \Rightarrow \sup\{\min\{\mu_A(y), \mu_B(z)\}\} \geq \alpha\)

\[
\Rightarrow \min\{\mu_A(a), \mu_B(b)\} \geq \alpha, \text{ for some } x = a + b.
\]

\[
\Rightarrow \mu_A(a) \geq \alpha \text{ and } \mu_B(b) \geq \alpha.
\]

\[
\Rightarrow v_A(a) \leq 1 - \mu_A(a) \leq 1 - \alpha = \beta \text{ and } v_B(b) \leq 1 - \mu_B(b) \leq 1 - \alpha = \beta.
\]

\[
\Rightarrow a \in C_{\alpha, \beta}(A), b \in C_{\alpha, \beta}(B).
\]

\[
\Rightarrow x = a + b \in C_{\alpha, \beta}(A) + C_{\alpha, \beta}(B).
\]

Thus \( C_{\alpha, \beta}(A + B) \subseteq C_{\alpha, \beta}(A) + C_{\alpha, \beta}(B) \) and so the equality follows.

**Remark 3.14.** In the above theorem the equality does not hold as can be seen by the following example: consider the ring \( R \) and IFI’s \( A \) and \( B \) of \( R \) as of the remark 3.9. Note that

\[
A + B = \{x, (\mu_A + \mu_B)(x), (v_A + v_B)(x) > | x \in R\}
\]

where

\[
(\mu_A + \mu_B)(x) = \sup\{\min\{\mu_A(a), \mu_B(b)\}\} \text{ and } (v_A + v_B)(x) = \inf\{\max\{v_A(a), v_B(b)\}\}
\]

\[
x = a + b
\]

Therefore we get

\[
(\mu_A + \mu_B)(0) = \sup\{\min\{\mu_A(0), \mu_B(0)\}\}, \min\{\mu_A(2), \mu_B(2)\}\}
\]

\[
= \sup\{\min\{0.8, 0.9\}, \min\{0.5, 0.4\}\} = 0.8
\]

next \( (\mu_A + \mu_B)(1) = \sup\{\min\{\mu_A(0), \mu_B(1)\}, \min\{\mu_A(1), \mu_B(0)\}, \min\{\mu_A(2), \mu_B(3)\}, \min\{\mu_A(3), \mu_B(2)\}\}
\]

\[
= \sup\{\min\{0.8, 0.3\}, \min\{0.2, 0.9\}, \min\{0.5, 0.3\}, \min\{0.2, 0.4\}\} = 0.3
\]

Similarly \( (\mu_A + \mu_B)(2) = 0.5 \) and \( (\mu_A + \mu_B)(3) = 0.3 \).
Similarly \((v_A + v_B)(0) = 0.1, (v_A + v_B)(1) = 0.6, (v_A + v_B)(2) = 0.2, (v_A + v_B)(3) = 0.6\) 

Now \(C_{5, 2}(A+B) = \{0, 2\}\) and from remark 3.9, we get \(C_{5, 2}(A) = \{0\}\) and \(C_{5, 2}(B) = \{0\}\). Therefore \(C_{5, 2}(A) + C_{5, 2}(B) = \{0\}\) ≠ \(\{0, 2\}\) = \(C_{5, 2}(A+B)\).

**Theorem 3.13.** For any two IFI’s \(A\) and \(B\) of a ring \(R\), 
\(C_{\alpha, \beta}(A)C_{\alpha, \beta}(B) \subseteq C_{\alpha, \beta}(AB)\) and the equality holds if \(\alpha + \beta = 1\).

**Proof.** We have \(AB = \{x, (\mu_A\mu_B)(x), (v_Av_B)(x) : x \in R\}\) where, 
\((\mu_A\mu_B)(x) = \inf \{\min \{\min \{\mu_A(a_i), \mu_B(b_i)\}\}\} \) and 
\(x = \sum a_ib_i \quad i < \infty\)

\[(v_Av_B)(x) = \inf \{\max \{\max \{v_A(a_i) v_B(b_i)\}\}\} \quad x = \sum a_ib_i \quad i < \infty\]

Let \(x = \sum a_ib_i \in C_{\alpha, \beta}(A)C_{\alpha, \beta}(B)\), where \(a_i \in C_{\alpha, \beta}(A)\) and \(b_i \in C_{\alpha, \beta}(B)\), for all \(i\).

Then \(\mu_A(a_i) \geq \alpha, v_A(a_i) \leq \beta,\) and \(\mu_B(b_i) \geq \alpha, v_B(b_i) \leq \beta,\) for all \(i\).

\(\implies \min \{\mu_A(a_i), \mu_B(b_i)\} \geq \alpha\) and \(\max \{v_A(a_i), v_B(b_i)\} \leq \beta,\) for all \(i\).

\(\implies \min \{\min \{\mu_A(a_i), \mu_B(b_i)\}\} \geq \alpha\) and \(\max \{\max \{v_A(a_i), v_B(b_i)\}\} \leq \beta\)

\(\implies \sup \{\min \{\mu_A(a_i), \mu_B(b_i)\}\} \geq \alpha\) and \(\inf \{\max \{v_A(a_i), v_B(b_i)\}\} \leq \beta\)

\(\implies (\mu_A\mu_B)(x) \geq \alpha\) and \(\inf (v_Av_B)(x) \leq \beta\)

\(\implies x \in C_{\alpha, \beta}(AB)\) and so the result follows.

For the second part let \(\alpha + \beta = 1\) and \(x \in C_{\alpha, \beta}(AB)\)

Then \((\mu_A\mu_B)(x) \geq \alpha\) and \((v_Av_B)(x) \leq \beta\)

Now \((\mu_A\mu_B)(x) \geq \alpha \implies \sup \{\min \{\min \{\mu_A(a_i), \mu_B(b_i)\}\}\} \geq \alpha\) and 
\(x = \sum a_ib_i \quad i < \infty\)

\(\implies \min \{\min \{\mu_A(a_i), \mu_B(b_i)\}\} \geq \alpha\) for some \(x = \sum a_ib_i\)

\(\implies \min \{\mu_A(a_i), \mu_B(b_i)\} \geq \alpha,\) for all \(i \implies \mu_A(a_i) \geq \alpha, \mu_B(b_i) \geq \alpha,\) for all \(i\).

\(\implies v_A(a_i) \leq 1 - \mu_A(a_i) \leq 1 - \alpha = \beta\) and \(v_B(b_i) \leq 1 - \mu_B(b_i) \leq 1 - \alpha = \beta,\) for all \(i\).

\(\implies a_i \in C_{\alpha, \beta}(A)\) and \(b_i \in C_{\alpha, \beta}(B),\) for all \(i\).

\(\implies x = \sum a_ib_i \in C_{\alpha, \beta}(A)C_{\alpha, \beta}(B)\)

\(i < \infty\)

Hence \(C_{\alpha, \beta}(AB) \subseteq C_{\alpha, \beta}(A)C_{\alpha, \beta}(B)\) and so the equality follows.

**Theorem 3.14.** If \(f : R \rightarrow R\) be an epimorphism and \(A = \{x, \mu_A(x), v_A(x) : x \in R\}\) be an IFI of \(R\) then \(f(C_{\alpha, \beta}(A)) \subseteq C_{\alpha, \beta}(f(A))\) and the equality holds if \(\mu_A\) has the sup property and \(\alpha + \beta = 1\).

**Proof.** Let \(y \in f(C_{\alpha, \beta}(A)),\) then \(y = f(x),\) where \(x \in C_{\alpha, \beta}(A)\). Then \(\mu_A(x) \geq \alpha, v_A(x) \leq \beta,\) where \(x \in f^{-1}(y) \implies \sup \{\mu_A(x) : x \in f^{-1}(y)\} \geq \mu_A(x) \geq \alpha\) and 
\(\inf\{v_A(x) : x \in f^{-1}(y)\} \leq v_A(x) \leq \beta \implies (f(\mu_A))(y) \geq \alpha \) and \((f(v_A))(y) \leq \beta\)
⇒ y ∈ C_{α, β}(f(A)). Hence the result follows.

For the other inclusion let μ_A has the sup property and α + β = 1. Let y ∈ C_{α, β}(f(A)), then (f(μ_A))(y) ≥ α and (f(ν_A))(y) ≤ β ⇒ Sup{μ_A(x) : x ∈ f^{-1}(y)} ≥ α and Inf{ν_A(x) : x ∈ f^{-1}(y)} ≤ β. Since μ_A has the sup property so there exists z ∈ f^{-1}(y) such that μ_A(z) = Sup{μ_A(x) : x ∈ f^{-1}(y)} ≥ α. Then ν_A(z) ≤ 1 − μ_A(z) ≤ 1 − α = β, so z ∈ C_{α, β}(A) and hence y = f(z) ∈ f(C_{α, β}(A)). Thus desired equality follows.

**Theorem 3.15.** If f : R → R' be a homomorphism and A = {<y, μ_A(y), ν_A(y)> : y ∈ R'} be an IFI of R' then f^1(C_{α, β}(A')) = C_{α, β}(f^{-1}(A')).

**Proof.** We have x ∈ f^{-1}(C_{α, β}(A')) ⇔ f(x) ∈ C_{α, β}(A') ⇔ μ_A(f(x)) ≥ α and ν_A(f(x)) ≤ β ⇔ (f^{-1}(μ_A))(x) ≥ α and (f^{-1}(ν_A))(x) ≤ β ⇔ x ∈ C_{α, β}(f^{-1}(A')). Hence the result follows.

**References**


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