Weakly Prime Subsemimodules of Semimodules over Semirings

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Abstract

The concept of weakly prime ideals in rings was introduced by D. D. Anderson and Eric Smith in 2003 [1]. Further, this concept in semirings and modules has been studied by many authors (see [2-5], [10]). We extend this concept for subsemimodule theory of semimodules over semirings. Indeed we prove: 1) If $M$ is an entire $R$-semimodule and $N$ is a weakly prime (weakly primary) subsemimodule of $M$, then $(N : M)$ is a weakly prime (weakly primary) ideal of $R$. 2) Let $N$ be a $k$-weakly prime ($k$-weakly primary) subsemimodule of an $R$-semimodule $M$. Then $N$ is either prime (primary) or $(N : M)N = 0$. Finally, we obtain some characterizations of weakly prime and weakly primary subsemimodules of semimodules over semirings.

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1. Introduction

Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D. D. Anderson and E. Smith in [1]. Here we study the weakly prime and weakly primary subsemimodules of semimodules over semirings. Some of our results are analogous to the results given in [2], [3], [4], [5] and [10]. Before we state some results let us introduce some notation and terminology.
For the definitions of monoid and semiring we refer [9]. All semirings in this paper are commutative with non-zero identity. $\mathbb{Z}_0^+ (\mathbb{N})$ will denote the set of all non-negative (positive) integers. An ideal $I$ of a semiring $R$ is called a subtractive ideal (= $k$-ideal) if $a, a + b \in I$, $b \in R$, then $b \in I$. A proper ideal $P$ of a semiring $R$ is said to be prime (primary) if $ab \in P$, then either $a \in P$ or $b \in P$ ($a \in P$ or $b^n \in P$ for some $n \in \mathbb{N}$). A proper ideal $P$ of a semiring $R$ is said to be weakly prime (weakly primary) if $0 \neq ab \in P$, then either $a \in P$ or $b \in P$ $(a \in P$ or $b^n \in P$ for some $n \in \mathbb{N}$) ([2],[5]). Radical of an ideal $I$ of a semiring $R$ is denoted by $radI$ and given by, $radI = \{ r \in R : r^n \in I \text{ for some } n \in \mathbb{N} \}$. Clearly $radI$ is an ideal of semiring $R$. Let $R$ be a semiring. A left $R$-semimodule is a commutative monoid $(M, +)$ with additive identity $0_M$ for which we have a function $R \times M \rightarrow M$, defined by $(r, x) \mapsto rx$ called scalar multiplication, which satisfies the following conditions for all elements $r$ and $r'$ of $R$ and all elements $x$ and $y$ of $M$:

1) $rr'x = r(r'x)$;
2) $r(x + y) = rx + ry$;
3) $(r + r')x = rx + r'x$;
4) $1_Rx = x$;
5) $r0_M = 0_M = 0_Rx$.

A nonempty subset $N$ of a left $R$-semimodule $M$ is called subsemimodule of $M$ if $N$ is closed under addition and closed under scalar multiplication. Throughout this paper, by an $R$-semimodule we mean a left semimodule over a semiring $R$. Every semiring $R$ is isomorphic to a $(\mathbb{Z}_0^+ , +, \cdot)$-semimodule ([9], P.151). An $R$-semimodule $M$ is said to be entire if $rm = 0$, $r \in R$, $m \in M$, then either $r = 0$ or $m = 0$. A subsemimodule $N$ of an $R$-semimodule $M$ is called subtractive subsemimodule (= $k$-subsemimodule) if $x, x + y \in N$, $y \in M$, then $y \in N$. If $N$ is a proper subsemimodule of an $R$-semimodule $M$, then we denote $(N : M) = \{ r \in R : rM \subseteq N \}$ and $rad(N : M) = \{ r \in R : r^nM \subseteq N \text{ for some } n \in \mathbb{N} \}$. Clearly $(N : M)$ and $rad(N : M)$ are ideals of $R$.

We need the following lemmas.

**Lemma 1.1.** ([8], Lemma 6) Let $I$ and $J$ be subtractive ideals of a semiring $R$. Then $I \cup J$ is subtractive ideal of $R$ if and only if $I \cup J = I$ or $I \cup J = J$.

**Lemma 1.2.** If $N$ is a subtractive subsemimodule of an $R$-semimodule $M$ and $m \in M$, then

1) $(N : M)$ is a subtractive ideal of $R$.
2) $(N : m) = \{ r \in R : rm \in N \}$ is a subtractive ideal of $R$.
3) $(0 : m) = \{ r \in R : rm = 0 \}$ is a subtractive ideal of $R$.
4) $(0 : M) = \{ r \in R : rm = 0 \text{ for all } m \in M \}$ is a subtractive ideal of $R$.

**Proof.** The proof is straightforward. \qed
2. WEAKLY PRIME AND WEAKLY PRIMARY SUBSEMIMODULES

In this section, we extend some definitions and results of S. E. Atani ([3], [4], [5]) to semimodules over semirings.

**Definition 2.1.** A subsemimodule $N$ of an $R$-semimodule $M$ is said to be prime (primary) if $rm \in N$, $r \in R, m \in M$, then either $r \in (N : M)$ or $m \in N$ ($r \in \text{rad}(N : M)$ or $m \in N$).

**Definition 2.2.** A subsemimodule $N$ of an $R$-semimodule $M$ is said to be weakly prime (weakly primary) if $0 \neq rm \in N$, $r \in R, m \in M$, then either $r \in (N : M)$ or $m \in N$ ($r \in \text{rad}(N : M)$ or $m \in N$).

Clearly, every prime subsemimodule of an $R$-semimodule is primary and weakly prime and hence weakly primary. Following examples show that the converse implications are not true.

**Example 2.3.** Let $R = (\mathbb{Z}_0^+, +, \cdot)$. Then

1) $4\mathbb{Z}_0^+$ is a primary subsemimodule of an $R$-semimodule $(\mathbb{Z}_0^+, +)$, which is not a prime subsemimodule.

2) $\{0\}$ is a weakly prime subsemimodule of an $R$-semimodule $(\mathbb{Z}_6, +_6)$, which is not a prime subsemimodule.

3) $\{0\}$ is a weakly primary subsemimodule of an $R$-semimodule $(\mathbb{Z}_6, +_6)$, which is not a primary subsemimodule.

4) $\{0, 4, 8\}$ is a weakly primary subsemimodule of an $R$-semimodule $(\mathbb{Z}_{12}, +_{12})$, which is not a weakly prime subsemimodule.

**Proposition 2.4.** Let $M$ be an entire $R$-semimodule and $N$ a weakly prime (weakly primary) subsemimodule of $M$. Then $(N : M)$ is a weakly prime (weakly primary) ideal of $R$.

**Proof.** Let $0 \neq ab \in (N : M)$ and $a \notin (N : M)$. Then there exists $0 \neq x \in M$ such that $ax \notin N$. Now $ax \neq 0$. Since $M$ is entire, $0 \neq abx \in abM \subseteq N$. Hence $0 \neq bax = abx \in N$. So $b \in (N : M)(b \in \text{rad}(N : M))$, since $N$ is a weakly prime (weakly primary) subsemimodule of $M$, as needed. \( \square \)

In the above proposition the condition that, $M$ is entire, is essential.

**Example 2.5.** Let $M = (\mathbb{Z}_6, +_6)$ and $R = (\mathbb{Z}_0^+, +, \cdot)$. Then $\{0\}$ is a weakly (weakly primary) subsemimodule of an $R$-semimodule $M$, but $\{\{0\} : M\} = 6\mathbb{Z}_0^+$ is not a weakly primary and hence not a weakly prime ideal because $0 \neq 2 \cdot 3 \notin 6\mathbb{Z}_0^+$, but $2 \notin 6\mathbb{Z}_0^+$, $3^n \notin 6\mathbb{Z}_0^+$ and $3 \notin 6\mathbb{Z}_0^+$, $2^n \notin 6\mathbb{Z}_0^+$ for all $n \in \mathbb{N}$.

**Theorem 2.6.** If $N$ is a weakly prime subtractive subsemimodule of an $R$-semimodule $M$, then either $N$ is prime or $(N : M)N = 0$.

**Proof.** Suppose $(N : M)N \neq 0$. Let $rm \in N$ with $r \in R$ and $m \in M$. If $rm \neq 0$, then we are through. Suppose $rm = 0$. If $rN \neq 0$ then there exists $n \in N$ such that $rn \neq 0$. Now $0 \neq r(m + n) = rn \in N$ implies either $r \in (N : M)$ or $m \in N$, as $N$ is a weakly prime subtractive subsemimodule.
Now suppose that $rN = 0$. If $(N : M)m \neq 0$, then there exists $r' \in (N : M)$ such that $r'm \neq 0$. So $0 \neq (r+r')m = r'm \in N$. Hence either $r + r' \in (N : M)$ or $m \in N$. By Lemma 1.2, $(N : M)$ is a subtractive ideal, and hence either $r \in (N : M)$ or $m \in N$. So suppose $(N : M)m = 0$. Since $(N : M)N \neq 0$, there exist $r'' \in (N : M)$ and $n' \in N$ such that $r''n' \neq 0$. Now $0 \neq r''n' = (r + r'')(m + n') \in N$. Since $N$ is a weakly prime subsemimodule, either $(r + r'') \in (N : M)$ or $(m + n') \in N$. Hence either $r \in (N : M)$ or $m \in N$, since $N$ is a subtractive subsemimodule and by Lemma 1.2, $(N : M)$ is a subtractive ideal. Hence $N$ is a prime subsemimodule of $M$.

**Lemma 2.7.** ([6], Lemma 3.1) Let $N$ be a proper subsemimodule of an $R$-semimodule $M$. Then the following statements are equivalent.

i) $N$ is a prime subsemimodule of $M$.

ii) If whenever $ID \subseteq N$, with $I$ is an ideal of $R$ and $D$ a subsemimodule of $M$, then $I \subseteq (N : M)$ or $D \subseteq N$.

**Theorem 2.8.** If $N$ is a proper subtractive subsemimodule of an $R$-semimodule $M$, then the following statements are equivalent.

i) If whenever $0 \neq ID \subseteq N$, with $I$ is an ideal of $R$ and $D$ a subsemimodule of $M$, then either $I \subseteq (N : M)$ or $D \subseteq N$.

ii) $N$ is a weakly prime subsemimodule of $M$.

iii) For $m \in M \setminus N$, $(N : m) = (N : M) \cup (0 : m)$.

iv) For $m \in M \setminus N$, $(N : m) = (N : M)$ or $(N : m) = (0 : m)$.

**Proof.** (i)$\Rightarrow$(ii) Suppose that $0 \neq rm \in N$ where $r \in R$ and $m \in M$. Take $I = Rr$ and $D = Rm$. Then $0 \neq ID \subseteq N$. So either $I \subseteq (N : M)$ or $D \subseteq N$ and hence either $r \in (N : M)$ or $m \in N$. Thus $N$ is a weakly prime subsemimodule. (ii)$\Rightarrow$(i) Suppose that $N$ is a weakly prime subsemimodule of $M$. If $N$ is prime, then the result is clear by using Lemma 2.7. So we can assume that $N$ is weakly prime that is not prime. Let $0 \neq ID \subseteq N$ with $x \in D \setminus N$. We show that $I \subseteq (N : M)$. Let $r \in I$. If $0 \neq rx \in ID \subseteq N$, then $r \in (N : M)$, as $N$ is a weakly prime subsemimodule. So assume that $rx = 0$. First suppose that $rD \neq 0$, say $rd \neq 0$ where $d \in D$. If $d \notin N$, then $r \in (N : M)$, as $N$ is a weakly prime subsemimodule. If $d \in N$, then $0 \neq r(d+x) = rd \in N$, so $r \in (N : M)$ or $d + x \in N$. Thus, $r \in (N : M)$, as $N$ is a subtractive subsemimodule. Hence $I \subseteq (N : M)$. So we can assume that $rD = 0$. Suppose that $Ix \neq 0$, say $ax \neq 0$ where $a \in I$. Then $a \in (N : M)$, as $N$ is a weakly prime subsemimodule. As $0 \neq (r + a)x = ax \in N$, we get $r \in (N : M)$, since $(N : M)$ is a subtractive ideal (by Lemma 1.2). So $I \subseteq (N : M)$. Therefore, we can assume that $Ix = 0$. Since $ID \neq 0$, there exist $b \in I$ and $d_1 \in D$ such that $bd_1 \neq 0$ and hence $0 \neq b(d_1 + x) = bd_1 \in N$. Since $N$ is a weakly prime subsemimodule, either $b \in (N : M)$ or $(d_1 + x) \in N$. If $b \notin (N : M)$ and $d_1 + x \in N$, then $d_1 \in N$ as $0 \neq bd_1 \in N$. Hence $x \in N$, since $N$ is a subtractive subsemimodule, which is a contradiction. If $b \in (N : M)$ and $d_1 + x \in N$, then $0 \neq bd_1 = b(d_1 + x) \in (N : M)N = 0$(By
Theorem 2.6), which is impossible. Hence $b \in (N : M)$ and $d_1 + x \notin N$. Since $0 \neq (r + b)(d_1 + x) = bd_1 \in N$, we obtain $r + b \in (N : M)$, so $r \in (N : M)$, since $(N : M)$ is a subtractive ideal (by Lemma 1.2). Hence $I \subseteq (N : M)$.

(ii)⇒(iii) Let $m \in M \setminus N$. Clearly $(N : M) \cup (0 : m) \subseteq (N : m)$. Now let $a \in (N : m)$. Then $am \in N$. If $am \neq 0$, then $a \in (N : M)$, since $N$ is a weakly prime subsemimodule. If $am = 0$, then $a \in (0 : m)$. Hence $(N : m) \subseteq (N : M) \cup (0 : m)$. (iii)⇒(iv) It follows by Lemma 1.1 and Lemma 2.10. (iv)⇒(ii) Suppose that $0 \neq rm \in N$ with $r \in R$ and $m \in M \setminus N$. Then $r \in (N : m)$ and $r \notin (0 : m)$. Hence $r \in (N : m) = (N : M)$, as required.

Theorem 2.9. Let $N$ be a proper $k$-subsemimodule of an $R$-semimodule $M$. Then the following statements are equivalent.

i) $N$ is a weakly primary subsemimodule of $M$.

ii) For $m \in M \setminus N$, $\text{rad}(N : m) = \text{rad}(N : M) \cup (0 : m)$.

Proof. (i)⇒(ii) Clearly, if $m \in M \setminus N$, then $\text{rad}(N : M) \cup (0 : m) \subseteq \text{rad}(N : m)$. Let $a \in \text{rad}(N : m)$ where $m \in M \setminus N$. Then $a^k m \in N$ for some $k \in \mathbb{N}$. If $a^k m \neq 0$, then $a \in \text{rad}(N : M)$, since $N$ is a weakly primary subsemimodule. If $a^k m = 0$, then assume that $s$ is the smallest positive integer with $a^s m = 0$. If $s = 1$, then $a \in (0 : m)$. Otherwise, $a \in \text{rad}(N : M)$, so $a \in \text{rad}(N : M) \cup (0 : m)$ and hence we have equality. (ii)⇒(i) Suppose that $0 \neq rm \in N$ with $r \in R$ and $m \in M \setminus N$. Then $r \in \text{rad}(N : m)$ and $r$ not in $(0 : m)$. It follows that $r \in \text{rad}(N : M)$, as required.

Lemma 2.10. Let $I$ be a subtractive ideal of a semiring $R$. If $a \in I$ and $a + b \in \text{rad}I$, then $b \in \text{rad}I$.

Proof. Let $a \in I, a + b \in \text{rad}I$. Therefore $(a + b)^m \in I$ for some $m \in \mathbb{N}$. So $c + b^m \in I$ for some $c \in I$. Since $I$ is a subtractive ideal, $b \in \text{rad}I$.

Theorem 2.11. If $N$ is a weakly primary subtractive subsemimodule of an $R$-semimodule $M$, then either $N$ is primary subsemimodule or $(N : M)N = 0$.

Proof. Suppose $(N : M)N \neq 0$. Let $rm \in N$ with $r \in R$ and $m \in M$. If $rm \neq 0$ then we are through. Suppose $rm = 0$. If $rN \neq 0$, then there exists $n \in N$ such that $rn \neq 0$. Now $0 \neq r(m + n) = rn \in N$ implies either $m \in N$ or $r \in \text{rad}(N : M)$, as $N$ is a weakly primary subtractive subsemimodule. Now suppose that $rN = 0$. If $(N : M)m \neq 0$, then there exists $r' \in (N : M)$ such that $r'm \neq 0$. So $(r + r')m \neq 0$ implies either $m \in N$ or $r + r' \in \text{rad}(N : M)$, By Lemma 1.2 and Lemma 2.10, either $m \in N$ or $r \in \text{rad}(N : M)$. So suppose $(N : M)m = 0$. Since $(N : M)N \neq 0$, there exist $r'' \in (N : M)$ and $n' \in N$ such that $r''n' \neq 0$. Now $0 \neq (r + r'')(m + n') \in N$. Since $N$ is a weakly primary subsemimodule, either $m + n' \in N$ or $r + r'' \in \text{rad}(N : M)$. If $m + n' \in N$, then $m \in N$ as $N$ is a subtractive subsemimodule of $M$. If $r + r'' \in \text{rad}(N : M)$, then $r \in \text{rad}(N : M)$, since by Lemma 2.10. Thus $N$ is a primary subsemimodule of $M$. 

\qed
3. WEAKLY PRIME AND WEAKLY PRIMARY SUBSEMIMODULES IN QUOTIENT SEMIMODULES

In this section, we extend results of S. E. Atani [2], to semimodules over semirings and give relation between the prime(secondary, weakly prime, weakly primary) subsemomodules of an \( R \)-semimodule \( M \) and the prime(secondary, weakly prime, weakly primary) subsemomodules of an \( R \)-semimodule \( M/N(Q) \)
where \( N \) is a partitioning subsemimodule of \( M \).

We need the following two Lemmas and two Theorems proved in [7].

**Lemma 3.1.** ([7], Lemma 2.2) Let \( N \) be a subsemimodule of an \( R \)-semimodule \( M \) and \( x, y \in M \) such that \( x + N \subseteq y + N \). Then \( x + z + N \subseteq y + z + N \) and \( rx + N \subseteq ry + N \) for all \( z \in M, r \in R \).

**Definition 3.2.** ([7]) A subsemimodule \( N \) of an \( R \)-semimodule \( M \) is called partitioning subsemimodule (= \( Q \)-subsemimodule) if there exists a subset \( Q \) of \( M \) such that
1) \( M = \cup \{ q + N : q \in Q \} \).
2) if \( q_1, q_2 \in Q \), then \( (q_1 + N) \cap (q_2 + N) \neq \emptyset \iff q_1 = q_2 \).

Let \( N \) be a partitioning subsemimodule of an \( R \)-semimodule \( M \). Then \( M/N(Q) = \{ q + N : q \in Q \} \) forms an \( R \)-semimodule under the following addition “\( + \)” and scalar multiplication “\( \circ \)” , \(( q_1 + N) \oplus (q_2 + N) = q_3 + N \) where \( q_3 \in Q \) is unique such that \( q_1 + q_2 + N \subseteq q_3 + N \), and \( r \circ (q_1 + N) = q_4 + N \) where \( q_4 \in Q \) is unique such that \( rq_1 + N \subseteq q_4 + N \). This \( R \)-semimodule \( M/N(Q) \) is called a quotient semimodule of \( M \) by \( N \) and denoted by \( (M/N(Q), +, \circ) \) or just \( M/N(Q) \).

**Lemma 3.3.** ([7], Lemma 3.4) Let \( N \) be a \( Q \)-subsemimodule of an \( R \)-semimodule \( M \). If \( A \) is a subtractive subsemimodule of \( M \) such that \( N \subseteq A \), then \( N \) is a \( Q \cap A \)-subsemimodule of \( A \).

**Theorem 3.4.** ([7], Theorem 3.5) Let \( N \) be a \( Q \)-subsemimodule of an \( R \)-semimodule \( M \). If \( A \) is a subtractive subsemimodule of \( M \) with \( N \subseteq A \), then \( A/N(Q \cap A) = \{ q + N : q \in Q \cap A \} \) is a subtractive subsemimodule of \( M/N(Q) \).

**Theorem 3.5.** ([7], Theorem 3.6) Let \( N \) be a \( Q \)-subsemimodule of an \( R \)-semimodule \( M \) and \( L \) a subtractive subsemimodule of \( M/N(Q) \). Then \( L = P/N(Q \cap P) \) for some subtractive subsemimodule \( P \) of \( M \) with \( N \subseteq P \).

**Lemma 3.6.** Let \( N \) be a \( Q \)-subsemimodule of an \( R \)-semimodule \( M \). If \( r \in R, m \in M \), then there exists a unique \( q \in Q \) such that \( rm \in r \circ (q + N) \).

**Proof.** Let \( r \in R, m \in M \). Since \( N \) is a \( Q \)-subsemimodule of \( M \) and \( m, rm \in M \), there exist unique \( q, q' \in Q \) such that \( m + N \subseteq q + N \) and \( rm + N \subseteq q' + N \). Also \( r \circ (q + N) = q'' + N \) where \( q'' \in Q \) is a unique element such that \( rq + N \subseteq q'' + N \). By Lemma 3.1, \( rm + N \subseteq rq + N \subseteq q'' + N \). Now \( rm \in (q' + N) \cap (q'' + N) \). Hence \( (q' + N) \cap (q'' + N) \neq \emptyset \). So \( q' = q''. \) Thus \( rm \in q' + N = q'' + N = r \circ (q + N) \).

\[\square\]
Theorem 3.7. Let $N$ be a $Q$-subsemimodule of an $R$-semimodule $M$ and $P$ a subtractive subsemimodule of $M$ with $N \subseteq P$. Then

1) If $P$ is a weakly primary subsemimodule of $M$, then $P/N_{(Q \cap P)}$ is a weakly primary subsemimodule of $M/N(Q)$.

2) If $N, P/N_{(Q \cap P)}$ are weakly primary subsemimodules of $M, M/N(Q)$ respectively, then $P$ is a weakly primary subsemimodule of $M$.

Proof. 1) Let $P$ be a weakly primary subsemimodule of $M$. Let $q_0$ be the unique element of $Q$ such that $q_0 + N$ is the zero element of $M/N(Q)([9], P.331)$. Let $r \in R, q_1 + N \in M/N(Q)$ be such that $q_0 + N \neq r \oplus (q_1 + N) \in P/N_{(Q \cap P)}$. By Lemma 3.3, there exists a unique $q_2 \in Q \cap P$ such that $r \oplus (q_1 + N) = q_2 + N$ where $rq_1 + N \subseteq q_2 + N$. Since $N \subseteq P$, $rq_1 \in P$. If $rq_1 = 0$, then $rq_1 \in (q_0 + N) \cap (q_2 + N)$, since $0 \in (q_0 + N)$ (by Lemma 2.3 [7]). So $q_0 = q_2$ and hence $q_0 + N = q_2 + N$ a contradiction. Thus $rq_1 \neq 0$. As $P$ is weakly primary subsemimodule, either $r^nM \subseteq P$ for some $n \in N$ or $q_1 \in P$. If $q_1 \in P$, then $q_1 \in Q \cap P$ and hence $q_1 + N \in P/N_{(Q \cap P)}$. Suppose $r^nM \subseteq P$. For $q + N \in M/N(Q)$, let $r^n \oplus (q + N) = q_3 + N$ where $q_3$ is a unique element of $Q$ such that $r^nq + N \subseteq q_3 + N$. Since $N \subseteq P$ and $P$ is a subtractive subsemimodule of $M$, $q_3 \in P$. Hence $q_3 \in Q \cap P$. Now $r^n \oplus (q + N) = q_3 + N \in P/N_{(Q \cap P)}$ and hence $r^n \oplus M/N(Q) \subseteq P/N_{(Q \cap P)}$. So $P/N_{(Q \cap P)}$ is a weakly primary subsemimodule of $M/N(Q)$.

2) Suppose that $N, P/N_{(Q \cap P)}$ are weakly primary subsemimodules of $M$, $M/N(Q)$ respectively. Let $0 \neq rm \in P$ where $r \in R, m \in M$. If $rm \in N$, then we are through, since $N$ is a weakly primary subsemimodule of $M$. Suppose $rm \in P \setminus N$. By using Lemma 3.6, there exists a unique $q_1 \in Q$ such that $m \in q_1 + N$ and $rm \in r \oplus (q_1 + N) = q_2 + N$ where $q_2$ is a unique element of $Q$ such that $rq_1 + N \subseteq q_2 + N$. Now $rm \in P, rm \in q_2 + N$ implies $q_2 \in P$, as $P$ is a subtractive subsemimodule and $N \subseteq P$. Hence $q_0 + N \neq r \oplus (q_1 + N) = q_2 + N \in P/N_{(Q \cap P)}$. As $P/N_{(Q \cap P)}$ is a weakly primary subsemimodule, either $r^n \oplus M/N(Q) \subseteq P/N_{(Q \cap P)}$ for some $n \in N$ or $q_1 + N \in P/N_{(Q \cap P)}$. If $q_1 + N \in P/N_{(Q \cap P)}$, then $q_1 \in P$. Hence $m \in q_1 + N \subseteq P$. So assume that $r^n \oplus M/N(Q) \subseteq P/N_{(Q \cap P)}$. Let $x \in M$. By using Lemma 3.6, there exists a unique $q_3 \in Q$ such that $x \in q_3 + N$ and $r^n \oplus (q_3 + N) = q_4 + N$ where $q_4$ is a unique element of $Q$ such that $r^nq_3 + N \subseteq q_4 + N$. Now $q_4 + N = r^n \oplus (q_3 + N) \in P/N_{(Q \cap P)}$ and hence $q_4 \in P$. As $r^n \oplus x \in q_4 + N$ and $N \subseteq P$ implies $r^n \oplus x \in P$. So $r^nM \subseteq P$. \hfill \Box

Theorem 3.8. Let $N$ be a $Q$-subsemimodule of an $R$-semimodule $M$ and $P$ a subtractive subsemimodule of $M$ with $N \subseteq P$. Then

1) If $P$ is a weakly prime subsemimodule of $M$, then $P/N_{(Q \cap P)}$ is a weakly prime subsemimodule of $M/N(Q)$.

2) If $N, P/N_{(Q \cap P)}$ are weakly prime subsemimodules of $M, M/N(Q)$ respectively, then $P$ is a weakly prime subsemimodule of $M$.

Proof. The proof is similar as in the proof of Theorem 3.7. \hfill \Box
**Theorem 3.9.** Let $N$ be a $Q$-subsemimodule of an $R$-semimodule $M$ and $P$ a subtractive subsemimodule of $M$ with $N \subseteq P$. Then $P$ is a prime (primary) subsemimodule of $M$ if and only if $P/N_{(Q \cap P)}$ is a prime (primary) subsemimodule of $M/N_{(Q)}$.

**Proof.** The proof is similar as in the proof of Theorem 3.7.

Every semiring $R$ is a semimodule over itself and hence every ideal $I$ of a semiring $R$ is a subsemimodule of an $R$-semimodule $R$. So we have

**Corollary 3.10.** Let $R$ be semiring, $I$ a $Q$-ideal of $R$ and $P$ a subtractive ideal of $R$ with $I \subseteq P$. Then $P$ is a prime (primary) ideal of $R$ if and only if $P/I_{(Q \cap P)}$ is a prime (primary) ideal of $R/I_{(Q)}$.

**REFERENCES**


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