Power groupoids and inclusion classes
of Abel Grassmann groupoids

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Abstract

Within the groupoid variety $AG = \left[(xy)z = (zy)x\right]_G$ of Abel Grassmann groupoids, we
determine the structure of all groupoids whose power groupoid is a band, a generalised inflation of
a band or a union of groups. Such groupoids are semigroup chains, inflations of semigroup trees or
$AG$ groupoids in the groupoid inclusion class $\left[\{(xy)z, x(yz)\} \subseteq \{x, y, z\}\right]_G$ respectively.

We prove that $G \in AG \cap \left[(xy)(zw) \in \{xy, xw, zw, yz\}\right]_G$ if and only if its power groupoid
$P(G)$ is an $AG$ inflation of an $AG$ band if and only if $P(G)$ is an $AG$ generalised
inflation of an $AG$ band.

1. Introduction

Groupoids in the (groupoid) variety $AG = \left[(xy)z = (zy)x\right]_G$ have been called Abel
Grassmann’s groupoids [10,11], left invertive groupoids [4] and LA- semigroups
[5]. In this paper we explore how the structure of the power groupoids of
groupoids in $AG$ can determine their structure. Within the variety $AG$ we show
that the collection of all groupoids whose power groupoid is a band [generalised
inflation of a band; a union of groups] determines an inclusion class of
$AG$ groupoids (cf. Theorems 1, 6 and 13). Similar work has been done with
semigroups in [3], [6], [8], [9] and [12].

Section 2 contains preliminary definitions, notation and some known results used
throughout the paper. In Section 3 we determine the structure of $AG$ groupoids in
which either the power groupoid or its square is a groupoid band. Such $AG$ groupoids are semigroup chains or inflations of semigroup trees
respectively. We also determine the structure of $AG$ groupoids whose power
groupoid is a union of groups. Results in this section are inspired by and build on the work of Redei [12], Pondelicek [9], Pelikan [8] and Monzo [6, 7].

2. Notation, definitions and preliminary results

By a groupoid we shall mean a set $G$ with a product $\ast: G \times G \to G$ and we shall denote $\ast(x, y)$ by $xy$. For $x \in G$ we define $x^1 \equiv x$ and, by induction, $x^n \equiv x^{n-1}x$ for any $n \in \{2, 3, \ldots\}$. For example, $x^4 = [(xx)x]x$. If $U^2 \subseteq U \subseteq G$ then $U$ is called a subgroupoid of $G$, which we denote by $U \leq G$. Also $G \cong H$ will denote that $G$ and $H$ are isomorphic groupoids.

A groupoid [semigroup] $G$ is an inflation of its subgroupoid [subsemigroup] $U$ if $G = \bigcup G_a (a \in U)$, where (1) $a \in G_a$ for every $a \in U$, (2) $G_a \cap G_b = \emptyset$ if $a \neq b$ and (3) for every $x \in G_a$ and $y \in G_b$, $xy = ab$. In this case, for any $x \in G_a (a \in U)$ we define $\hat{x} \equiv a$, and so $xy = \hat{x}y$ for any $\{x, y\} \subseteq G$.

A groupoid [semigroup] $G$ is a [symmetric] generalised inflation of its subgroupoid [subsemigroup] $U$ if (1) $a \in G_a$ for every $a \in U$, (2) $G_a \cap G_b = \emptyset$ if $a \neq b$, (3) for every $x \in G_a$ there exist $\alpha_x$ and $\beta_x$, right and left mappings respectively on $U$, such that $xy = ba_x \beta_y a$ for every $x \in G_a$ and $y \in G_b$ and (4) $a \in U$ implies $\alpha_a = \kappa_a = \beta_a$, where $\kappa_a$ is a constant mapping that sends every element to $a$. [5] $\alpha_a = \beta_a (x \in U)$

An element $a \in G$ is called idempotent [3-potent] if $a = a^2$ [and $a^2 = aa^2$]. The groupoid $G$ is called a groupoid band [3-potent] if every element of $G$ is idempotent [3-potent]. Then $E_G = \{x \in G : xx = x\}$ is the set of idempotents of $G$.

If in addition a groupoid band is commutative then it will be called a groupoid semilattice.

If $\{a, a'\} \in G$ and $a = (aa')a = a(a')$ then $a$ is called a regular element of $G$ and $a'$ is called a partial inverse of $a$. If, in addition, $a' = (a'a)a = a'(aa')$ then $a'$ is called an inverse of $a$. In this case $a$ and $a'$ are called mutual inverses. The set of all partial inverses [inverses] of the element $a$ is denoted by $P(a)[V(a)]$. Then $P_c(a) = \{a' \in P(a) : aa' = a'a\}$ and $V_c(a) = \{a' \in V(a) : aa' = a'a\}$. If $P_c(a) \neq \emptyset$ then $a$ is called a *-regular element of $G$. Also $P_r(a) = \{a' \in G : (aa')a = a\}$ and $P_l(a) = \{a' \in G : a(a'a) = a\}$ are the sets of left and right partial inverses of $a$. 


The groupoid $G$ is called a regular groupoid [inverse groupoid] if every element of $G$ is regular [has a unique inverse]. We will use REG [INV] to denote the collection of all regular [inverse] groupoids.

If $G$ satisfies the equation $(xy)z = (zy)x$ then $G$ is called an $AG$ groupoid. We use $AG$ to denote the collection of all $AG$ groupoids, $SEM$ to denote the collection of all semigroups and $UG$ to denote the collection of all groupoids that are a union of (not necessarily disjoint) groups. The power groupoid of a groupoid $G$ is defined as $\{ AB : A , B \subseteq G \}_{\neq} \ $ where $AB = \{ ab : a \in A , b \in B \}$. A chain $S$ is a semigroup in which, for any $\{ x,y \} \subseteq S$, either $x = xy = yx$ or $y = xy = yx$. The collection of all semigroup chains will be denoted by $C$. If $H_i \subseteq K_i (i \in \{ 1,2,...,n \})$ is a finite collection of inclusions, where $H_i , K_i (i \in \{ 1,2,...,n \})$ are meaningful groupoid {semigroup} words over some alphabet, then $[H_i \subseteq K_i ;...;H_n \subseteq K_n] \subseteq \{ [H_i \subseteq K_i ;...;H_n \subseteq K_n] \}$ denotes the collection of all groupoids {semigroups} that satisfy the inclusions $H_i \subseteq K_i (i \in \{ 1,2,...,n \})$ and will be called an inclusion class of groupoids {semigroups}.

If $I$ is any collection of groupoids then $INF (I)$ will denote the collection of all inflations of groupoids in $I$. A groupoid $G$ is called a groupoid semilattice if $G \subseteq \{ x = x^2 ; xy = yx \}_{\neq}$. A groupoid semilattice is called a groupoid tree if $e = eg = ge$ and $f = fg = gf$ implies $ef = fe \in \{ e,f \}$. If a groupoid semilattice [groupoid tree] $G$ is also a semigroup then we call $G$ a semilattice [tree]. We denote the collection of all (semigroup) trees by $T$. The collection of all 3-potent $AG$ groupoids $G \subseteq \{ x = x^2 x \}_{\neq}$ will be denoted as $AG_2$. The following results are well known and will be used throughout this paper.

**Result 1.** [5] A commutative $AG$ groupoid is a semigroup.

**Result 2.** [5] An $AG$ groupoid with a right identity element is commutative (and is therefore a semigroup).

**Result 3.** [5] If $G \in AG$ then for any $\{ x,y,z,w \} \subseteq G$, $(xy)(zw) = (xz)(yw)$.

**Result 4.** If $G \in AG$ then $E_G \leq G$. If $E_G \in \{ xy \in \{ x,y \} \}_{\neq}$ then $E_G \in C$. 

Result 5. [11] Let $G \in \mathbf{AG}$. Then $G \in \left[ x = x^2, x = xx^2 \right]_G$ if and only if $G$ is an $\mathbf{AG}$ band $Y = E_G$ of abelian groups of order 2.

Result 6. [3,12] $S \in \left[ xy \in \{x, y\} \right]$ iff $S$ is a chain $Y$ of semigroups $S_\alpha \in \left[ xy = x \right] \cup \left[ xy = y \right]$ ($\alpha \in Y$), where $x = xy = yx$ whenever $x \in S_\alpha, y \in S_\beta$ and $\alpha = \alpha \beta = \beta \alpha$.

Result 7. [1] $S \in \mathbf{SEM}$ is an inflation of a band iff $S \in \left[ xy = x^2y^2 \right]$.

Result 8. [2] $S \in \mathbf{SEM}$ is a generalised inflation of a band iff $S \in \left[ xy = xyxy \right]$.

Result 9. [13] If $S \in \mathbf{SEM}$ and $S^2$ is a semilattice of groups then $S \in \mathbf{INF} \left( \{S^2\} \right)$.

Result 10. $G \in \mathbf{AG}$ iff $P \left( G \right) \in \mathbf{AG}$.

Result 11. [10] If $G \in \mathbf{AG}$ then the following statements are equivalent: (1) $G$ is a generalised inflation of an $\mathbf{AG}$ band; (2) $G$ is an inflation of an $\mathbf{AG}$ band; (3) $G \in \left[ xy = x^2y^2 \right]_G$ and (4) $G^2 \in \left[ x = x^2 \right]_G \cap \mathbf{AG}$.

Result 12. If $G \in \mathbf{AG}$ is a generalised inflation of $U \leq G$ then $G^2 \cong U^2$.

Result 13. [7, Cor. 16] If $G \in \mathbf{AG}$ and $G^2 \in \mathbf{UG}$ then $G$ is an inflation of $G^2$.

Result 14. [7, Theorem. 12] If $G \in \mathbf{AG}$ then $G \in \mathbf{UG}$ iff $\forall x \in G, \left| P_i \left( x \right) \right| = 1$.

3. Power groupoids and inclusion classes of $\mathbf{AG}$ groupoids

Clearly a semigroup $S \in \left[ xy \in \{x, y\} \right]$ if and only if $P \left( S \right) \in \left[ x = x^2 \right]$. Such semigroups are chains of left or right-zero semigroups (cf. [12], p.181, [3] and [6], 4.2). In [9] it was proved that a semigroup $S \in \left[ xyzw \in \{xy, xw, zw, zw, yz\} \right]$ if and only if $P \left( S \right)^2 \in \left[ x = x^2 \right]$ and the structure of such semigroups was completely described. With groupoids, $P \left( G \right) \in \left[ x = x^2 \right]_G$ if and only if $G \in \left[ xy \in \{x, y\} \right]_G$ and $P \left( G \right)^2 \in \left[ x = x^2 \right]_G$ if and only if $G \in \left[ (xy)(zw) \in \{xy, xw, zw, zw\} \right]_G$. We
Power groupoids and inclusion classes

95

proceed to find all $AG$ groupoids $G$ in which $P \ (G)$ is a groupoid band and those in which $[P \ (G)]^2$ is a groupoid band.

Theorem 1. $[xy \in \{x, y\}_G \cap AG = C$

Proof: It is easy to see that any groupoid $G$ contained in the intersection of the two groupoid inclusion classes is idempotent. Let $\{x, y, z\} \subseteq G$. Then $xy = x^2y = (yx)x = yx$ and so $G$ is commutative. By Result 1, $G$ is a commutative semigroup. Since $G \in \{xy \in \{x, y\}_G \}$ and $G$ is commutative, either $x = xy = yx$ or $y = xy = yx$; therefore $G \in C$. Conversely, if $G \in C$ then clearly $G \in \{xy \in \{x, y\}_G \}$. Since $G$ is commutative, $G \in AG$.

Theorem 2. $[(xy)(zw) \in \{xy, xw, zy, zw\}_G \cap AG = INF \ (T)$

Proof. Let $G \in \{(xy)(zw) \in \{xy, xw, zy, zw\}_G \cap AG = \}$. Let $\{e, f, g\} \subseteq E_G$. Then $fe = (ef) = (ef)(ff) \in \{ef, f\}$ and so $ef \in \{fe, e\}$. Therefore either $ef = fe$ or $fe = f$ and $ef = e$. But $ef = (fe)e$ and so either $ef = fe$ or $e = ef = (fe)e = fe = f$. So $E_G$ is commutative and therefore is a semigroup. Furthermore, if $e = eg = ge$ and $f = fg = gf$ then $fe = ef = (eg)(fg) \in \{eg, fg\} = \{e, f\}$ and $E_G$ is a tree. Also, since for any $\{a, b, c\} \subseteq G$, $(ab)(ab) \in \{ab\}$, $E_G = G^2$. So $ab = (ab)^2 = a^2b^2$ and $(ab)c = (av)^2c^2 = \left[a^2+b^2\right]^2c^2 = (a^2b^2)c^2 = a^2\left(b^2c^2\right) = a^2\left(b^2c^2\right)^2 = a(bc)$ and so $G \in \{xy = x^2y^2\}$. By Result 9, $G \in INF \ (T)$ and we have shown that $[(xy)(zw) \in \{xy, xw, zy, zw\}_G \cap AG = INF \ (T)$. Conversely, suppose that $G \in INF \ (T)$. Then, since an inflation of a commutative semigroup is commutative, $G \in AG$. Then since $G^2 \in T$, by Lemma 4.6 [6], $G^2 \in \{(xy)(zw) \in \{xy, xw, zy, zw\}_G \}$. Clearly, since $G$ is an inflation of $G^2$, $G \in \{(xy)(zw) \in \{xy, xw, zy, zw\}_G \} \subseteq \{(xy)(zw) \in \{xy, xw, zy, zw\}_G \}$. So, $INF \ (T) \subseteq \{(xy)(zw) \in \{xy, xw, zy, zw\}_G \} \cap AG$.

Lemma 3. If $G \in AG$ then $P \ (G)$ is an inflation of an $AG$ union of groups iff $P \ (G)$ is a generalised inflation of a union of groups iff $[P \ (G)]^2$ is a union of groups.
Proof. Let $G \in AG$. By definition it is easy to see that if $P(G)$ is an $AG$ inflation of a union of groups then $P(G)$ is an $AG$ generalised inflation of a union of groups, which by Result 12 implies $[P(G)]^2$ is a union of groups, which by Results 10 and 13 implies $P(G)$ is an $AG$ inflation of a union of groups. □

Lemma 4. $G \in AG \cap \big[(xy)(zw) \in \{xz, xw, yz, yw\}\big]_G$ iff $P(G)$ is an $AG$ inflation of an $AG$ groupoid band.

Proof. $(\Rightarrow)$ Using Results 10 and 11, we must show that $P(G) \in \big[xy = x^2y^2\big]_G$. Let $\{a,a'\} \subseteq A \subseteq G$ and $\{b,b'\} \subseteq B \subseteq G$. Then $(aa')(bb') \in \{ab\}$ and so $AB \subseteq A^2B^2$. Also, $(aa')(bb') \in \{ab, ab', a'b, a'b'\} \subseteq AB$ and so $A^2B^2 \subseteq AB \subseteq A^2B^2$. Hence, $P(G) \in \big[xy = x^2y^2\big]_G$. $(\Leftarrow)$ By Result 11, $P(G) \in \big[xy = x^2y^2\big]_G$. Let $\{a,b,c,d\} \subseteq G$ and let $A = \{a,b\}$ and $B = \{c,d\}$. Then $(ab)(cd) \in A^2B^2 = AB = \{ac, ad, bc, bd\}$ and so $G \in \big[(xy)(zw) \in \{xz, xw, yz, yw\}\big]_G$. Since $P(G) \in AG$, by Result 10, $G \in AG$. □

Lemma 5. $G \in AG \cap \big[(xy)(zw) \in \{xy, xw, yz, zw\}\big]_G$ iff $P(G)$ is an $AG$ generalised inflation of an $AG$ groupoid band.

Proof. $(\Rightarrow)$ Using Lemma 3 and Results 10 and 11, we need only show that $[P(G)]^2 \in \big[x = x^2\big]_G$. As in the proof of Theorem 2, $G \in \big[xy = x^2y^2\big]_G$. Therefore, for any $A \subseteq G$ and $B \subseteq G$, $AB \subseteq A^2B^2$. Also, for any $\{a,a'\} \subseteq A$ and $\{b,b'\} \subseteq B$, $(aa')(bb') = (ab)(a'b') \in \{ab, ab', a'b, a'b'\} \subseteq AB$. Hence, $A^2B^2 \subseteq AB \subseteq A^2B^2$ and $AB = A^2B^2 = (AB)^2$ and $[P(G)]^2 \in \big[x = x^2\big]_G$. $(\Leftarrow)$ By Lemma 3 and Results 10 and 11, $[P(G)]^2 \in \big[x = x^2\big]_G$. Let $\{a,b,c,d\} \subseteq G$ and let $A = \{a,c\}$ and $B = \{b,d\}$. Then $(ab)(cd) \in (AB)^2 = AB = \{ab, ad, cb, cd\}$. So, $G \in \big[(xy)(zw) \in \{xy, xw, yz, zw\}\big]_G$. Since $P(G) \in AG$, by Result 10, $G \in AG$. □

Theorem 6. The following statements are equivalent:

(1) $G \in AG \cap \big[(xy)(zw) \in \{xy, xw, yz, zw\}\big]_G$
Power groupoids and inclusion classes

(2) \( G \in AG \cap [(xy)(zw) \in \{xz,xw,yz,yw\}]_g \);
(3) \( G \in INF (T) \);
(4) \( P(G) \) is an \( AG \) inflation of an \( AG \) band;
(5) \( P(G) \) is an \( AG \) generalised inflation of an \( AG \) band and
(6) \( G \in AG \) and \( [P(G)]^2 \) is a groupoid band.

We proceed to determine the structure of groupoids in other inclusion classes of \( AG \) groupoids.

Lemma 7. \([xy \in \{x^3, y^3\}] = INF ([xy \in \{x, y\}])\).

Proof: Note that we are dealing with inclusion classes of semigroups, so that the products are associative. Suppose that \( S \in INF ([xy \in \{x, y\}]) \). Then \( S \) is an inflation of a band and so, by Result 7, \( S \in [xy = x^2y^2] \). But \( S^2 \in [xy \in \{x, y\}] \) and therefore \( xy = x^2y^2 \in \{x^2, y^2\} \). Hence,

\( INF ([xy \in \{x, y\}]) \subseteq [xy \in \{x^2, y^2\}] \). However, since \( S \) is an inflation of a band, for every \( a \in S \), \( a^2 = \alpha^2 = \alpha = \alpha^3 = \alpha^3 \) for some idempotent \( \alpha \). Therefore, \( INF ([xy \in \{x, y\}]) \subseteq [xy \in \{x^3, y^3\}] \). Now suppose that \( S \in [xy \in \{x^3, y^3\}] \).

Then \( a^2 = \alpha^3 (a \in S) \) and so \( E_S = S^2 = \{a \in S : a = a^2\} \in [xy \in \{x, y\}] \). For any \( \{a, b\} \subseteq S \), if \( ab = a^2 \) then \( a^2b^2 = a(ab)b = a^3b = \alpha^2(ab) = \alpha^4 = a^2 = ab \).

Similarly, if \( ab = b^2 \) then \( a^2b^2 = ab \). Therefore, \( S \in [xy = x^2y^2] \) and so \( S \) is an inflation of the band \( S^2 \in [xy \in \{x, y\}] \). We have shown that \( INF ([xy \in \{x, y\}]) \subseteq [xy \in \{x^3, y^3\}] \subseteq INF ([xy \in \{x, y\}]) \) as required. ■

Theorem 8. The following statements are equivalent:

(1) \( G \in INF (C) \)
(2) \( G \in INF ([xy \in \{x, y\}] \cap [xy = yx]) = INF ([xy \in \{x, y\}]_g \cap AG) \)
(3) \( G \in [xy \in \{x^2x, y^2y\}]_g \cap AG \)
(4) \( G \in [xy \in \{x^2x, y^2y\}]_g \cap AG \)
(5) \( G \in [xy \in \{x^3, y^3\}] \cap [xy = yx] \).
Proof. It follows easily from Result 4 that 
\[ I \cap N \cap F (C) = I \cap N \cap F \left( \left[ xy \in \{x, y\} \right] \cap \left[ xy = yx \right] \right) \]. It then follows easily from Theorem 1 that (1) and (2) are equivalent. It is also easy to see that 
\[ I \cap N \cap F (C) \subseteq \left[ xy \in \{x^2, y^2\} \right] \cap \left[ xy \in \{x^2, y^2\} \right] \cap \left[ G \cap A \right] \], so that (1) implies (3) and (4). Assume (4). For \( a \in G \), \( a^2 = a^2 a = \left( a^2 \right)^2 \in E_g \). For \( \{e, f\} \subseteq E_g \), \( e f = e^2 e = e \) and so \( e = e f = (e f) e = e f = f \) and \( E_g = \{0\} \). But then for any \( \{a, b, c, d\} \subseteq G \), \( ab = a^2 a = a^2 = c^2 = c^2 c = cd \). Thus, \( E_g = G^2 = \{0\} \). But since \( ab = a^2 = a^2 b^2 = 0 \), \( G \) is an inflation of \( G^2 = \{0\} \), where \( x = 0 \) \( (x \in G) \). So 
\[ xy = (xy)^2 = x^2 y^2 \]. By Result 7, \( G \in I \cap N \cap F (C) \) and so (4) implies (1). Similarly, (3) implies (1). Hence, (1), (2), (3) and (4) are equivalent. Then, using Lemma 7, (1) through (5) are equivalent. $\blacksquare$

Theorem 9. \( \left[ xy = x^2 x \right]_G \cap \left[ G \cap A \right] = I \cap N \cap F \left( \{0\} \right) \)

Proof. Let \( a \in G \in \left[ xy = x^2 x \right]_G \cap \left[ G \cap A \right] \). Then \( a^2 = a^2 a = \left( a^2 \right)^2 \in E_g \). For any \( \{e, f\} \subseteq G \), \( e f = e^2 e = e \) and so \( e = e f = (e f) e = e f = f \) and \( E_g = \{0\} \). But then for any \( \{a, b, c, d\} \subseteq G \), \( ab = a^2 a = a^2 = c^2 = c^2 c = cd \). Thus, \( E_g = G^2 = \{0\} \). But since \( ab = a^2 = a^2 b^2 = 0 \), \( G \) is an inflation of \( G^2 = \{0\} \), where \( x = 0 \) \( (x \in G) \). So 
\[ \left[ xy = x^2 x \right]_G \subseteq I \cap N \cap F \left( \{0\} \right) \]. It is straightforward to show that 
\( I \cap N \cap F \left( \{0\} \right) \subseteq \left[ xy = x^2 x \right]_G \). $\blacksquare$

Theorem 10. \( \left[ (xy) x \in \{x, y\} \right]_G \cap \left[ G \cap A \right] \subseteq AG \)

Proof. Let \( a \in G \in \left[ (xy) x \in \{x, y\} \right]_G \cap \left[ G \cap A \right] \). Then \( a^2 = (a^2 a) a = \left( a^2 \right)^2 \) and \( a a^2 = (a^2 a) a^2 \in \{a^2, a\} \). So either \( a a^2 = a \) or \( a a^2 = (a a^2) a^2 = \left( a^2 \right)^2 a = a^2 a = a \). So \( a^2 a = a = a a^2 \) and \( G \in AG \). $\blacksquare$

Lemma 11. Let \( G \in AG \) and \( (G) \in REG \) . Then \( E_G \neq \emptyset \) implies \( E_G \in C \).

Proof. Let \( A = \{e, f\} \subseteq E_G \). Then there exists \( B \subseteq G \) such that \( (AB) A = A (BA) = A \). Let \( b \in B \). Then \( (eb) f \in (AB) A = A = \{e, f\} \).

CASE 1: \((eb) f = e\). Note that \( e (be) e = A (BA) = A = \{e, f\} \) and so
\[ e = (eb) f = (fb) e = (fe)(be) = \left( (fe)(be) \right) e = \left( e(\{fe\}, f(\{fe\}) \right) e \in \{e(\{fe\}, f(\{fe\}) \right) \}.
\]
So, \( e = (eb) f = (ef)(bf) \) and \( ef = \left( (ef)(bf) \right) f = \left[ f (bf) \right] (ef) \in \{e(\{ef\}, f(\{ef\}) \right) \} \) since \( f (bf) \in A(BA) = A = \{e, f\} \).

**CASE 1.1 :** \( ef = e(ef) \). Note that \( e = (eb) f = \left[ (eb) f \right] e = (ef)(eb) = e(fb) \). Therefore, \( ef = e(ef) = e\left[ (eb) f \right] = e\left[ f (eb) \right] = e\left( (fe)(fb) \right) =
\[ e \in \{e(\{fe\}, f(\{fe\}) \right) e \} . \]
But, since \( e \in \{e(\{fe\}, f(\{fe\}) \right) \} \), we can assume that \( e = f (fe) \) or else \( ef = e f = e \). Now \( eb = \left[ f (fe) \right] b = \left[ b(f e) \right] f \) and so
\[ f (eb) = f \left[ \left( b(f e) \right) \right] f = \left( \left( fb \right) f (fe) \right) f = f = \left( f (eb) \right) \] Then, \( e = ee = \left[ f (fe) \right] \left[ e(f b) \right] = \left( f (fe) \right) \left[ f (fe) \right] = \left( fe \right) (ef) \), so
\[ fe = f \left( \left( fe \right) (ef) \right) \] Since \( e \in \{e(\{fe\}, f(\{fe\}) \right) \} \), we can assume that \( e = f (fe) \) or else \( ef = (fe)e = e \). Then, \( e = (eb) f = (fb) = (fe)(be) \) and
\[ so \quad fe = f \left[ \left( fe \right)(be) \right] = \left[ f (fe) \right] \left[ f (be) \right] =
\[ \left( fe \right) \left[ f (be) \right] \in \{\left( fe \right) f, \left( fe \right)e\} = \{ef\} \] So \( e = (ef)e = (fe)e = ef \).

**CASE 2 :** \( (eb) f = f \). Then
\[ f = (eb) f = (fb) e = (fe)(be) = \left[ \left( fe \right)(be) \right] f = \left[ f (be) \right] (fe) \in \{e(\{fe\}, f(\{fe\}) \right) \} \] So \( fe = \left[ (eb) f \right] e = \left[ (eb) e \right] (fe) = \left( fe \right) \left( eb \right) e \in \{\left( fe \right) e, \left( fe \right) f \} \).

**CASE 2.1 :** \( fe = (fe) f = f (ef) \). Note that \( fe = (fe) f = \left[ \left( fe \right) f \right] f = f (fe) \). If \( f = f (fe) \) then \( fe = f (fe) = f \) , which implies that \( ef = (fe)e = fe = f \) . So we can assume that \( f = e(\{fe\}) = (ef)e \). Then
\[ f = ff = \left[ e(\{fe\}) \right] f = \left[ f (fe) \right] e = (fe)e = ef \] .

**CASE 2.2 :** \( fe = (fe)e = ef \). Then, if \( f = e(\{fe\}) = (ef)e \) ,
\[ ef = fe = \left[ \left( ef \right) e \right] e = e(ef) = e(f e) = f \] and so we can assume that \( f = f (fe) \).
Then, \( f = f (fe) = \left[ f (fe) \right] (fe) = (fe) f = (ef) f = fe \), which implies that
$ef = (ef)^2 = (fe)^2 = f^2 = f$. So we have shown that $ef \in \{e, f\}$ and, by Result 4, if $E_G \neq \emptyset$ then $E_G \in C$. ■

**Theorem 12.** $G \in AG$ and $P(\langle G \rangle) \in UG$ implies $G \in C$ or $G = E_g \cup \{q\}$ where $E_G \in C$, $q \notin E_G$, $qe = eq = e$ for every $e \in E_G - \{q^2\}$ and $q^2x = xq^2 = x$ for every $x \in G$.

**Proof.** First we will prove that every element of $G$ has a partial inverse with which it commutes, which by Result 14 implies that $G \in UG$.

Let $a \in G$, with $A = \{a\}$. Then since $P(\langle G \rangle) \in UG$ there exists $B \subseteq G$ such that $(AB)A = A(AB) = A$ and $AB = BA = (AB)^2 = (BA)^2$. Let $b \in B$.

Then $(ab)a = a(ba) = a$ and $(ba)a = (ab)(ba) = (ab)(ba)^2 = (ba)^2(ba) = (ab)(ba)$ and so $(ba)a = a$ and $ab = (ba)^2$. Also, $a^2a = a^2[(ba)a] = [a(ba)]a^2 = aa^2$. So for any $x \in G$ we can show similarly that $x^2x = xx^2$. Then $ab = [(ab)a]b = (ba)(ab) = (ba)(ba)^2 = (ba)^2(ba) = (ab)(ba)$ and so $ab = [(ab)(ba)](ba) = (ba)^2(ab) = (ab)^2$. Similarly, there exists $b' \in G$ such that $b = (bb')b$ and $bb' \in E_G$. Then, $ba = (ba)(bb') \in E_G$ and so $ab = (ba)^2 = ba$.

Hence, $G \in UG$ and by Lemma 11, $\emptyset \neq E_g \in C$.

We now show that for any $a \in G, a^2 \in E_g$. Let $A = \{a, a^2\}$ and let $B \in P(\langle G \rangle)$ be such that $B \in V_e(A)$. If $b \in B$ then $(a^2b)a = [(ba)a]a \in (AB)A = A = \{a, a^2\}$. But $(ba)a \in (BA)A = (AB)A = A = \{a, a^2\}$ and so $(a^2b)a \in \{a, a^2\}$. If $(a^2b)a = a^2$ then $a^2 = (a^2b) \in \{a, a^2\}$, which implies that $a^2 \in E_g$. If $(a^2b)a = a$ then $a^2 = a^2(a^2b)$. But $a^2b = (ba)a \in \{a, a^2\}$ and so $a^2 = a^2(a^2b) \in \{a^2, a^2a, (a^2)^2\}$, which implies that $a^2 \in E_g$.

We now prove that for any $\{x, e\} \subseteq G$ satisfying $x \neq x^2 \neq e \in E_g$, $e = ex = xe$ and $e = x^2e = ex^2$. Let $A = \{x, e\}$ and let $B \in V_e(A)$, with $b \in B$. Then $(eb)x \in \{x, e\}$. Suppose that $(eb)x = x$. Now $(eb)e = (be)e \in \{x, e\}$. But this implies that $eb = e$, or else $x = (eb)x = x^2$, a contradiction. Hence, $x = (eb)x = ex$. Then $xb = (ex)b = (bx)e \in \{x, e\}$, which implies $xb = x$, or else
\[ x = (eb)x = (xb)x = e = e^2 = x^2. \]
Then \[ x^2 = (xb)x \in \{x, e\} \], a contradiction. So we can assume that \((eb)x = e\). Now \(eb = (be)e \in \{x, e\} \), which implies that \(eb = e\) or else \(e = (eb)x = x^2\), a contradiction. So \(e = (eb)x = ex\). Now \(xe = x(eb) \in \{x, e\} \), which implies \(xe = e\) or else \(e = ex = (xe)e = xe = x\), a contradiction. So we have shown that \(e = ex = xe\). (As a consequence, \(e = e^2 = x^2e\).)

We now prove that \(|G - E_g| \leq 1\). Let \(\{x, y\} \subseteq G - E_g\), with \(x \neq y\). We have already proved that \(\{x^2, y^2\} \subseteq E_g\) and we know that \(x \neq x^2\) and \(y \neq y^2\). In the paragraph above we proved that if \(x^2 \neq y^2\), then \(y^2 = y^2x = xy^2\) and \(x^2 = x^2y = y^2x\). Therefore, \(y^2 = y^2x = (y^2x)x = x^2y^2\) and similarly \(x^2 = y^2x^2\). But this is a contradiction, since \(E_g \in C\) implies that \(x^2y^2 = y^2x^2\). So \(x^2 = y^2\), which means that any two non-idempotent elements of \(G\) are in the set \(\{g \in G : g^2 = a\}\), where \(a \in E_g\) and \(a\) is maximal in \(E_g\). We have already shown that \(G \in UG\) and since \(e = xe = ex\) for every \(e \in E_g\) (when \(e \neq x^2\)) and \(x \neq x^2\), \(x = xx^2 = x^2x\) for every \(x \in G_u - E_g\). But then it is easy to see that \(G_u\) is an abelian group of order two. Thus, \(\{x, y\} \subseteq G_u\), with \(a = 1_{G_u}\). Take \(A = \{a, x, y\}\) and let \(B \subseteq G\) satisfy \(B \in V_c(A)\). Then for \(b \in B\), \((ab)a \in \{a, x, y\}\). This implies that \(b^2 = a\) and hence, \(b \in G_u\). If \((ab)a = a\) then \(b = a\) and therefore \((xb)y = xy \in (AB)A = \{a, x, y\}\). If \((ab)a = x\) then \(b = x\) and \(xy = (ax)y = (ab)y \in \{a, x, y\}\). If \((ab)a = y\) then \(b = y\) and \(xy = (xy)a = (xb)a \in \{a, x, y\}\). We have therefore shown that \(xy \in \{a, x, y\}\). But \(xy = a\), implies \(x = y\), \(xy = x\) implies \(y = a\) and \(xy = y\) implies \(x = a\). Hence we have a contradiction to the hypothesis that \(\{x, y\} \subseteq G - E_g\) and \(x \neq y\) and therefore \(|G - E_g| \leq 1\).}

**Theorem 13.** \(G \in AG\) and \(P(G) \in UG\) if and only if \(G \in \{\{(xy)z, x(yz)\} \subseteq \{x, y, z\}\}_G \cap AG\)

**Proof.** \((\Rightarrow)\) It follows easily from Theorem 12 that \(G \in \{\{(xy)z, x(yz)\} \subseteq \{x, y, z\}\}_G \cap AG\).

\((\Leftarrow)\) Let \(\{e, f\} \subseteq E_g\). Then \(ef = (fe)e \in \{e, f\}\), so by Result 4, \(E_g \in C\). Note that for any \(z \in G\), \(z = z^2z = zz^2\) and \(z^2 = (z^2)z = (z^2)^2 \in E_g\). Assume that
\{x, y\} \subseteq G - E_g and x \neq y. Then (xy) y = y^2 x = (y^2 y^2) x \in \{x, y\} \cap \{x, y\}$. Since \(x \neq y\) and \(y \neq y^2\), (xy) y = x. Similarly, (yx) x = y. But then 
\[y = (yx) x = \left(\left(\left(x y\right) x\right) x\right) x = \left(\left(x y\right) x\right) x - x^2 y x \cap \{x, y, x\} \cap \{x, y, x, x\}.\]
But \(y \notin \{x, x^2\}\) and so \(y = xy = yx\). By symmetry, \(x = yx = xy\) and so \(x = y\). We have shown that either \(G = E_g \in C\) or \(G = E_g \cup \{q\}\), where \(q \notin E_g\). Clearly, \(qq^2 = q^2 q = q\). Let \(e \in E_g\). Then \(q^2 e \in E_g\) and \(q^2 e = (q^2 q^2) e \in \{e, q\}\), which implies that \(q^2 e = e\). Similarly, \(eq^2 = e\). So we have shown that \(q^2 x = xq^2 = x\) for all \(x \in G\). If \(e \neq q^2\) then \(qe = q(ee) \in \{q, e\}\). If \(qe = q\) then \(q = q(e) e = eq\) and then \(q^2 = (qe) q \in \{e, q\}\), a contradiction. Hence, \(qe = e\). Similarly, \(eq = e\). We have shown that \(G\) is commutative and so \(G \in SEM\). For any \(A \in P\) \((G)\), clearly \(A(AA) = (AA) A \subseteq A\). Then \(A \subseteq (AA) A = A(AA)\) if \(q \notin A\). If \(q \in A\) then, since \(q = q^2 q = qq^2\), it is still the case that \(A \subseteq (AA) A = A(AA)\). Then by Result 5 and 10, \(P\) \((G) \in UG\).

Note that in the proof of Theorem 13 we have also proved the following.

**Corollary 14.** If \(G \in AG\) and \(G\) has the form described in Theorem 12 then \(P\) \((G) \in UG \cap \left[x = x^2 x = xx^2\right]_g\).

**Corollary 15.** \(G \in \left[\{xy\} z, x (yz)\right] \subseteq \{x, y, z\} \in AG\) implies that \(G\) has the form described in Theorem 12.

**Lemma 16.** Let \(G \in AG\). Let \(S_n\) be the following statement: Any product \(P\) of \(2n+1\) factors \((a_1, a_2, ..., a_{2n+1}\) say) in \(G\) is equal to a product \(P'\) of the same factors in which \(P'\) contains a product of the form \(a bc\) or \((ab)c\), where \(\{a, b, c\} \subseteq \{a_1, a_2, ..., a_{2n+1}\}\). Then \(S_n\) is true for every positive integer \(n\).

**Proof.** (By induction on \(n\).) Clearly, \(S_1\) is true. Let \(n \geq 2\) and assume that \(S_n\) is true for all \(t \leq n-1\). Let \(P\) be any product, in \(G\), of the \(2n+1\) factors \(a_1, a_2, ..., a_{2n+1}\) listed in order of their appearance in \(P\).

**CASE 1:** \(P\) begins in \((a_1 a_2) P'\). If \(P'\) has an odd number of factors and, by the induction hypothesis, \(P'\) (and, therefore, \(P\)) contains a product of the required form. If \(P'\) has an even number of factors then \(P\) begins in
(a_1a_2)P' = (P'a_2)a_1, and by the induction hypothesis P'a_2 contains a product of the required form, so that P is equal to a product containing a product of the required form.

**Case 2:** If $P$ begins in $a_1P' = a_i(AB)$ We can assume that $A$ and $B$ both have an even number of factors. We can assume that $A$ has at least four factors, or else $AB = (a_2a_3)B = (Ba_3)a_2$, which by the induction hypothesis contains a factor of the required form. Then $A = A_1B_1$ and again we can assume that $A_1$ has an even number of factors, but not two factors, and $B_1$ also has an even number of factors. Continuing in this manner we eventually obtain a factor $A_1B_1 = (B_1a_{k+1})a_k$ and $B_1a_{k+1}$ has a factor of the required form. Hence $P$ is equal to a product of the $a_j$'s $(j \in \{1, 2, \ldots, 2n+1\})$ and this product has a factor of the required form.

**Lemma 17.** Let $G \in AG$ and $n \in \mathbb{N} = \{1, 2, \ldots\}$. Let $A_n$ be the following statement: Any product of $n$ factors equals one of the factors. Then for any $n \in \mathbb{N}$, $A_2$ is valid if and only if $A_{2n}$ is valid and $A_3$ is valid if and only if $A_{2n+1}$ is valid.

**Proof.** $[A_{2n} \Rightarrow A_2]$ Let $\{a, b\} \subseteq G$. Then by $A_{2n}$, $a^{2n} = a$. So, again by using $A_{2n}$, $a^2a = a^2a^{2n} \in \{a, a^2\}$. This implies that $a^2 = (a^2)^2$. Then $a^2 = (a^2)^n$, which by $A_{2n}$ equals $a$. That is, $a = a^2$ for any $a \in G$. Using this fact and applying $A_{2n}$ again gives $ab = (ab)^n \in \{a, b\}$ and so $A_2$ is valid.

$[A_2 \Rightarrow A_n]$, for any $n \in \mathbb{N}$, $n \geq 3$] Let $\{a, b, c\} \subseteq G$ and assume that $A_3$ is valid. Then $(ab)c \in \{ab, c\} \subseteq \{a, b, c\}$ and $a(bc) \in \{a, bc\} \subseteq \{a, b, c\}$. So $A_3$ implies $A_3$. Assume that $A_t$ is valid for all $t \in \mathbb{N}$, $3 \leq t \leq n-1$. Let $\{a_1, a_2, \ldots, a_q, a_{q+1}, \ldots, a_p\}$ be the factors of an arbitrary product $T = AB$, where $T$ has $n$ factors, $a_1, a_2, \ldots, a_q$ are the factors of $A$ and $a_{q+1}, a_{q+2}, \ldots, a_p$ are the factors of $B$. Since, by the induction hypothesis, $A_2, A_q$ and $A_{n-q}$ are valid, $T \in \{A, B\} \subseteq \{a_1, a_2, \ldots, a_n\}$ and so $A_n$ is valid.

Note that we have proved that for any $n \in \mathbb{N}$, $A_{2n}$ implies $A_2$ implies $A_{2n}$.

$[A_{2n+1} \Rightarrow A_3]$, for any $n \in \mathbb{N}$] Let $a \in G$. Then $a^2a = a^2a^{2n+1} \in \{a, a^2\}$ which implies that $a^2 \in E_G$. Then since $a^2a = (a^2)^n = a = a(a^2)^n = aa^2$, by Result 5
$G$ is a union of groups of order 2. Let $\{u, v, w\} \subseteq G$. Then

$$(uv)w = (uv)(ww^2) = (uv) \left[ w \left( w^3 \right)^{w^{-1}} \right] \in \{u, v, w\}.$$ and similarly, $u(vw) \in \{u, v, w\}$. Hence, $A_3$ is valid.

[ $A_3 \Rightarrow A_{2n+1}$, for any $n \in \mathbb{N}$.] Using Lemma 16, the proof of this by induction is straightforward.

**Theorem 18.** If $G \in AG$ then the following statements are equivalent.

1. $P(G) \in UG$;
2. $G \in C$ or $G = E_G \cup \{q\}$ where $E_G \in C$, $q \notin E_G$, $qe = eq = e$ ($q^2 \neq e \in E_G$) and $q^2 x = xq^2$ ($x \in G$);
3. $G \in \left[ \{x(yz), (xy)z\} \subseteq \{x, y, z\} \right]_{AG}$
4. $P(G) \in \left[ x^2 x = x^2 \right]_{AG}$
5. $G \in AG$ and satisfies $A_{2n+1}$ for some $n \in \mathbb{N}$ and
6. $G \in AG$ and satisfies $A_{2n+1}$ for every $n \in \mathbb{N}$.

**Proof.** [ $1 \iff 3$ ] This is Theorem 13.

[ $1 \iff 2$ ] This follows from Theorem 12 and Corollary 14.

[ $1 \iff 5 \iff 6$ ] This follows from Lemmas 16 and 17.

[ $2 \Rightarrow 4$ ] This follows from Corollary 14 and Result 10.

[ $4 \Rightarrow 1$ ] This follows from Result 5. ■

**Open Questions**

1. Does there exist a symmetric generalised inflation $S$ of a commutative semigroup $U$ such that $S$ is not an inflation of $U$?

2. Does there exist a symmetric generalised inflation $G \in AG$ of a groupoid $U \in AG$ such that $G$ is not an inflation of $U$?

3. If $G \in AG$ does $P(G) \in REG$ imply $P(G) \in UG$?
References


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