An Introduction to Γ-Semirings

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Abstract
This paper is an introduction of Γ-semirings. We first consider the congruences and ideals of a Γ-semiring. Then we construct a new Γ-semiring and discuss the formation of ideals on this Γ-semiring. Also, by using the congruences induced by homomorphisms of a Γ-semiring, we establish some isomorphism theorems and investigate the commutativity of some diagrams. In particular, some fundamental theorems of Γ-semirings are proved and strengthened.

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1 Introduction

Γ-semirings were first studied by M. K. Rao [26] as a generalization of Γ-ring as well as of semiring. It is noted that Γ-rings were considered by N. Nobusawa in 1964 in [25], there have been a few slightly different definitions for a Γ-ring. The concepts of Γ-semirings and its sub-Γ-semirings with a left(right) unity was studied by J. Luh [23] and M. K. Rao in [26]. The ideals, prime ideals, semiprime ideals, $k$-ideals and $h$-ideals of a Γ-semiring, regular Γ-semiring, respectively, were extensively studied by S. Kyuno [21] and M. K. Rao [26].

In Γ-semirings, the properties of their ideals, prime ideals, semiprime ideals and their generalizations play an important role in their structure theory, however
the properties of an ideal in semirings and \( \Gamma \)-semirings are somewhat different from the properties of the usual ring ideals (see [1-4, 10-17, 22, 23]). In order to amend these differs, the concepts of \( k \)-ideals and \( h \)-ideals in a semiring were introduced and considered by D. R. LaTorre [22] in 1965. For the properties of some \( h \)-ideals in \( \Gamma \)-semirings, the reader is referred to the recent papers of T. K. Dutta and S. K. Sardar, K. P. Shum in [7-10] and [30]. The notions of operator semirings and \( \Gamma \)-semirings were introduced by them. Moreover, the properties of prime ideal[8] and semiprime ideals in a \( \Gamma \)-semirings were studied and discussed by them [9]. It is noted here that the theory of \( \Gamma \)-semiring has been enriched with the help of operator semirings of a \( \Gamma \)-semiring by Dutta and Sardar [7]. In order to make the operator semirings more effective in the study of \( \Gamma \)-semirings, T. K. Dutta and S. Sardar [7] established the correspondence between the ideals of a \( \Gamma \)-semiring \( R \) and the ideals of the operator semirings of \( R \).

The notion of \( \Gamma \)-semiring not only generalizes the notions of semiring and \( \Gamma \)-ring but also the notion of ternary semiring. We point out here that this notion provides an algebraic background to the non-positive cones of the totally ordered rings. We recall here that the non-negative cones of the totally ordered rings form semirings but the non-positive cones do not form semirings because the induced multiplication is no longer closed. For further study of semirings, \( \Gamma \)-semirings and their generalization, the reader is referred to [5, 6, 15-20, 23, 24].

In this paper, by using the congruences induced by homomorphisms, we reestablish some fundamental isomorphism theorems and investigate the commutativity of some diagrams of \( \Gamma \)-semirings.

2 Preliminaries

We first recall some concepts which will be used throughout this paper.

Let \((R, +)\) and \((\Gamma, +)\) be commutative semigroups. Then we call \(R\) a \(\Gamma\)-semiring if there exists a map \(R \times \Gamma \times R \to R\), written \((x, \gamma, y)\) by \(x\gamma y\), such that it satisfies the following axioms for all \(x, y, z \in R\) and \(\gamma, \beta \in \Gamma\):

1. \(x\gamma(y + z) = x\gamma y + x\gamma z\) and \((x + y)\gamma z = x\gamma z + y\gamma z\);
2. \(x(\gamma + \beta)y = x\gamma y + x\beta y\);
3. \((x\gamma y)\beta z = x\gamma(y\beta z)\).

In the above case, we call \(R = (R, \Gamma)\) a \(\Gamma\)-semiring.
A Γ-semiring $R$ is said to have a zero element if there exists an element $0 \in R$ such that $0 + x = x = x + 0$ and $0 \gamma x = 0 = x \gamma 0$ for all $x \in R$ and $\gamma \in \Gamma$. Also, a Γ-semiring $R$ is said to be commutative if $x \gamma y = y \gamma x$, for all $x, y \in R$ and $\gamma \in \Gamma$.

A subsemigroup $I$ of $R$ is called an ideal of $(R, \Gamma)$, if $I \Gamma R \subseteq I$ and $R \Gamma I \subseteq I$, where by $I \Gamma R$ we mean the set \{\$x \gamma r \mid x \in I, r \in R, \gamma \in \Gamma\$. If $R$ is a Γ-semiring with zero element, then it is easy to verify that every ideal $I$ of $(R, \Gamma)$ has the zero element.

**Example 2.1.** (1) Let $S_1$ and $S_2$ be two additive commutative semigroups. Let $S$ be the additive commutative semigroup of all homomorphisms from $S_1$ to $S_2$ and $\Gamma$ be the additive commutative semigroup of all homomorphisms from $S_1$ to $S_2$. Then $S$ is a Γ-semiring, where $a \gamma b$ denotes the usual composition of homomorphisms, $a, b \in S$ and $\gamma \in \Gamma$.

(2) Let $R = (\mathbb{Z}^+, +)$ be the semigroup of positive integers and let $\Gamma = (2\mathbb{Z}^+, +)$ be the semigroup of even positive integers. Then, $R$ is a Γ-semiring.

(3) Let $R$ be the additive commutative semigroup of all $m \times n$ matrices over the set of all non-negative integers and $\Gamma$ be the additive commutative semigroup of all $n \times m$ matrices over the same set. Then, we can verify that $R$ is a Γ-semiring, where $a \gamma b$ is the usual matrix product, for any $a, b \in R$ and $\gamma \in \Gamma$.

Let $R_1$ be a $\Gamma_1$-semiring and $R_2$ a $\Gamma_2$-semiring. Then $(\varphi, g) : (R_1, \Gamma_1) \rightarrow (R_2, \Gamma_2)$ is called a homomorphism if $\varphi : R_1 \rightarrow R_2$ and $g : \Gamma_1 \rightarrow \Gamma_2$ are homomorphisms of semigroups such that $\varphi(x \gamma y) = \varphi(x)g(x)\varphi(y)$ for all $x, y \in R_1$ and $\gamma \in \Gamma_1$. The mapping $(\varphi, g)$ is called an epimorphism if $(\varphi, g)$ is a homomorphism and $\varphi$ and $g$ are epimorphisms of semigroups. Similarly, we can define a monomorphism. A homomorphism $(\varphi, g)$ is an isomorphism if $(\varphi, g)$ is an epimorphism and a monomorphism.

### 3 Ideals of a Γ-semiring

Throughout this paper, $R$ will be a Γ-semiring unless otherwise specified.

The following lemmas are easy to prove.

**Lemma 3.1.** Let $\Lambda$ be a non-empty index set and $\{I_\lambda\}_{\lambda \in \Lambda}$ be a family of ideals of $(R, \Gamma)$. Then, $\bigcap_{\lambda \in \Lambda} I_\lambda$ is an ideal of $(R, \Gamma)$. 
**Lemma 3.2.** Let $\mathcal{L}(R, \Gamma)$ be the set of all ideals of $(R, \Gamma)$. Then $(\mathcal{L}(R, \Gamma), \subseteq, \wedge, \vee)$ is a complete lattice, where $I \wedge J = I \cap J$ and $I \vee J = \langle I \cup J \rangle$ is the unique smallest ideal containing $I \cup J$.

The following theorem is a theorem of quotient $\Gamma$-semiring and a correspondence theorem.

**Theorem 3.3.** Let $R$ be a $\Gamma$-semiring with zero and $I$ an ideal of $(R, \Gamma)$. Then $R/I = \{x + I \mid x \in R\}$ is a $\Gamma$-semiring with the mapping

$$
* : \frac{R}{I} \times \Gamma \times \frac{R}{I} \longrightarrow \frac{R}{I},
$$

defined by

$$(x + I) * \gamma \ast (y + I) = x \gamma y + I,$$

for all $x, y \in R$ and $\gamma \in \Gamma$.

**Proof.** We first define an operation $\oplus$ on $\frac{R}{I}$ by:

$$(x + I) \oplus (y + I) = x + y + I,$$

for all $x + I, y + I \in \frac{R}{I}$. It is easy to see that $\oplus$ and $*$ are well-defined. Consequently, we can verify that $(\frac{R}{I}, \oplus)$ is a commutative semigroup and we have the following equalities:

$$(x + I) * \gamma \ast (y + I) \oplus (z + I) = (x + I) * \gamma \ast (y + z + I) = x \gamma (y + z) + I = x \gamma y + x \gamma z + I = (x \gamma y + I) \oplus (x \gamma z + I) = (x + I) * \gamma \ast (y + I) \oplus (x + I) * \gamma \ast (z + I).$$

Similarly, we have

$$
((x + I) \oplus (y + I)) * \gamma \ast (z + I) = (x + I) * \gamma \ast (z + I) \oplus (y + I) * \gamma \ast (z + I),
$$
Ideals, congruences, homomorphisms and isomorphisms

\[(x + I) \ast (\gamma + \beta) \ast (y + I) = (x + I) \ast \gamma \ast (y + I) \oplus (x + I) \ast \beta \ast (y + I),\]

\[\left( (x + I) \ast \gamma \ast (y + I) \right) \ast \beta \ast (z + I) = (x + I) \ast \gamma \ast \left( (y + I) \ast \beta \ast (z + I) \right).\]

These show that \(\frac{R}{I}\) is a \(\Gamma\)-semiring.

\[\square\]

**Theorem 3.4.** (Correspondence Theorem) Let \(R\) be a \(\Gamma\)-semiring with zero and \(J\) an ideal of \(R\) such that \(I \subseteq J\). Then \(\frac{J}{I}\) is an ideal of \(\left( \frac{R}{I}, \Gamma \right)\). Conversely, If \(K\) is an ideal of \(\left( \frac{R}{I}, \Gamma \right)\), then there exists an ideal \(J\) of \((R, \Gamma)\) such that \(I \subseteq J\) and \(K = \frac{J}{I}\).

The proof of this theorem is routine and we hence omit the proof.

4 Commutative \(\Gamma\)-semirings and congruences on a \(\Gamma\)-semiring

In this section, we consider a commutative \(\Gamma\)-semiring \(R\). We have the following theorems.

**Theorem 4.1.** The following conditions on an ideal \(I\) of a commutative \(\Gamma\)-semiring \(R\) with zero are equivalent:

1. \(H + (0 : I) = (H \Gamma I : I)\) for all ideals \(H\) of \((R, \Gamma)\).
2. \(H \Gamma I = K \Gamma I\) implies that \((0 : I) + H = (0 : I) + K\) for all ideals \(H\) and \(K\) of \((R, \Gamma)\),

where \((I : A) = \{x \in R \mid x \gamma a \in I, \text{ for all } a \in A \text{ and } \gamma \in \Gamma\}\) for any \(\emptyset \neq A \subseteq R\).

**Theorem 4.2.** Let \(R\) be a commutative \(\Gamma\)-semiring. If \(I\) is an ideal of \((R, \Gamma)\), \(\phi \neq A \subseteq R\) and \(\gamma \in \Gamma\) then the following statements hold:

1. \(I \subseteq (I : A) \subseteq (I : A \Gamma A) \subseteq (I : A \gamma A)\) for all \(\gamma \in \Gamma\).
2. If \(A \subseteq I\), then \((I : A) = R\).

**Theorem 4.3.** Let \(R\) be a commutative \(\Gamma\)-semiring. If \(I\) is an ideal of \((R, \Gamma)\), \(\phi \neq A \subseteq R\). Then
\[(I : A) = \bigcap_{a \in A} (I : a) = (I : A \setminus I).\]

An equivalence relation \(\theta\) on \((R, \Gamma)\) is said to be a congruence if for all \(x, y, z \in R\) and \(\gamma \in \Gamma\), we have

\[
x \theta y \implies (x + z) \theta (y + z),
\]
\[
x \theta y \implies (x \gamma z) \theta (y \gamma z) \text{ and } (z \gamma x) \theta (z \gamma y).
\]

By \(R : \theta\), we mean the set of all equivalence classes of the elements of \(R\) with respect to the mapping \(\theta\), that is,

\[R : \theta = \{\theta(x) \mid x \in R\}.
\]

**Lemma 4.4.** Let \(\theta\) be a congruence relation on \((R, \Gamma)\). Then

\[
\theta(x + y) = \theta\left(\theta(x) + \theta(y)\right) \text{ and } \theta(x \gamma y) = \theta\left(\theta(x) \gamma \theta(y)\right)
\]

for all \(x, y \in R\) and \(\gamma \in \Gamma\).

**Proof.** We first observe that \(\theta(x + y) \subseteq \theta\left(\theta(x) + \theta(y)\right)\) and \(\theta(x \gamma y) \subseteq \theta\left(\theta(x) \gamma \theta(y)\right)\). By routine checking, we can easily verify that the above equalities hold. \(\square\)

In the next theorem, we demonstrate how to construct a new \(\Gamma\)-semirings by using the congruence relations.

**Theorem 4.5.** Let \(\theta\) be a congruence on \((R, \Gamma)\). Define \(\oplus\) on \(R : \theta\) by \(\theta(x) \oplus \theta(y) = \theta(x + y)\) for all \(x, y \in R\). Then, \((R : \theta, \oplus)\) is a \(\Gamma\)-semiring with the following map:

\[
\odot : (R : \theta) \times \Gamma \times (R : \theta) \longrightarrow (R : \theta),
\]

defined by \(\theta(x) \odot \gamma \odot \theta(y) = \theta(x \gamma y)\), for all \(x, y \in R\) and \(\gamma \in \Gamma\).
Proof. Let \( \theta(x) = \theta(x') \) and \( \theta(y) = \theta(y') \). Then, by Lemma 4.4, we have the following equality.

\[
\theta(x) \oplus \theta(y) = \theta(x + y) = \theta\left(\theta(x) + \theta(y)\right) = \theta\left(\theta(x') + \theta(y')\right) = \theta(x') \oplus \theta(y').
\]

Also, we have an additional equality

\[
\theta(x) \odot \gamma \odot \theta(y) = \theta(x\gamma y) = \theta\left(\theta(x)\gamma\theta(y)\right) = \theta\left(\theta(x')\gamma\theta(y')\right) = \theta(x') \odot \gamma \odot \theta(y').
\]

Thus, \( \oplus \) and \( \odot \) are well-defined.

Hence, we can verify that \( (R: \theta, \oplus) \) is a commutative semigroup. Now, we deduce that

\[
\theta(x) \odot \gamma \odot \left(\theta(y) \oplus \theta(z)\right) = \theta(x) \odot \gamma \odot \theta(y + z) = \theta(x\gamma(y + z)) = \theta(x\gamma y + x\gamma z) = \theta(x\gamma y) \oplus \theta(x\gamma z) = \left(\theta(x) \odot \gamma \odot \theta(y)\right) \oplus \left(\theta(x) \odot \gamma \odot \theta(z)\right).
\]

Similarly, we can prove that

\[
\left(\theta(x) \oplus \theta(y)\right) \odot \gamma \odot \theta(z) = \left(\theta(x) \odot \gamma \odot \theta(z)\right) \oplus \left(\theta(y) \odot \gamma \odot \theta(z)\right),
\]

\[
\theta(x) \odot (\gamma + \beta) \odot \theta(y) = \left(\theta(x) \odot \gamma \odot \theta(y)\right) \oplus \left(\theta(x) \odot \beta \odot \theta(y)\right),
\]

\[
\left(\theta(x) \odot \gamma \odot \theta(y)\right) \odot \beta \odot \theta(z) = \theta(x) \odot \gamma \odot \left(\theta(y) \odot \beta \odot \theta(z)\right).
\]

Therefore, \( R: \theta \) is a \( \Gamma \)-semiring. \( \square \)

Lemma 4.6. If \( \Pi_R : R \rightarrow R : \theta \) is defined by \( \Pi_R(x) = \theta(x) \) and \( 1_\Gamma \) is the identity function on \( \Gamma \), then \( (\Pi_R, 1_\Gamma) : (R, \Gamma) \rightarrow (R : \theta, \Gamma) \) is an epimorphism.

Proof. Let \( x, y \in R \) and \( \gamma \in \Gamma \). Then, it is easy to see that
\[ \Pi_R(x + y) = \theta(x + y) = \theta(x) \oplus \theta(y) = \Pi_R(x) \oplus \Pi_R(y). \]

Also, we have

\[ \Pi_R(x \gamma y) = \theta(x \gamma y) = \theta(x) \odot \gamma \odot \theta(y) = \Pi_R(x) \odot 1_\Gamma(\gamma) \odot \Pi_R(y). \]

Clearly, \( \Pi_R \) is surjective and \((\Pi_R, 1_\Gamma)\) is an epimorphism.

\section{Congruences and products of \(\Gamma\)-semirings}

In this section, we show how to use an ideal and a congruence on a \(\Gamma\)-semiring \(R\) to construct a new ideal of \(R\) and to investigate the relationship between them.

**Theorem 5.1.** Let \(\theta\) be a congruence on \((R, \Gamma)\). If \(I\) is an ideal of \((R, \Gamma)\), then \(C_I = \{x \in R \mid x \theta a, \exists a \in I\}\) is an ideal of \((R, \Gamma)\) and \(I \subseteq C_I\).

**Proof.** It is clear that \(I \subseteq C_I\). Let \(x, y \in C_I\). Then \(x \theta a\) and \(y \theta b\) for some \(a, b \in I\). On the other hand, \(\theta\) is a congruence on \(R\) which implies that \((x + y) \theta (a + b)\) and \(x + y \in C_I\). Now, let \(x \in C_I\), \(r \in R\) and \(\gamma \in \Gamma\). Then, \(x \theta a\) for some \(a \in I\). In other words, \(\theta\) is a congruence on \(R\) which implies that \((x \gamma r) \theta (a \gamma r)\). Thus, \(x \gamma r \in C_I\). Similarly, we can prove that \(r \gamma x \in C_I\). Therefore, \(C_I\) is an ideal of \((R, \Gamma)\). \(\square\)

By using the standard arguments, we can prove the following theorem.

**Theorem 5.2.** Let \(R\) be a \(\Gamma\)-semiring with zero and \(\theta\) a congruence on \((R, \Gamma)\). Then, \(\theta(0)\) is an ideal of \((R, \Gamma)\).

In the next two theorems, we state the connections between the ideals of \((R, \Gamma)\) and \((R : \theta, \Gamma)\).

**Theorem 5.3.** If \(I\) is an ideal of \((R, \Gamma)\), then \(I : \theta\) is an ideal of \((R : \theta, \Gamma)\).

**Theorem 5.4.** If \(J\) is an ideal of \((R : \theta, \Gamma)\), then there exists an ideal \(I\) of \((R, \Gamma)\) such that \(J = I : \theta\).
Proof. Define $I = \{ x \in R \mid \theta(x) \in J \}$. Then we have

$$
\theta(x) \in J \implies x \in I \implies \theta(x) \in I : \theta,
$$

and

$$
\theta(x) \in I : \theta \implies \exists a \in I, \theta(x) = \theta(a) \implies \theta(x) = \theta(a) \in J.
$$

Thus, $J = I : \theta$.

Now, suppose that $x, y \in I$. Then $\theta(x), \theta(y) \in J$ and by Theorem 4.5, we have $\theta(x + y) = \theta(x) \oplus \theta(y) \in J$. Hence, $x + y \in I$. Also, assume that $x \in I$, $r \in R$ and $\gamma \in \Gamma$. Then, we have $\theta(x) \in J$ and by Theorem 4.5, we have $\theta(x\gamma r) = \theta(x) \odot \gamma \odot \theta(r) \in J$. Hence, $x\gamma r \in I$. Similarly, we can prove that $r\gamma x \in I$. Therefore, $I$ is an ideal of $(R, \Gamma)$.

Lemma 5.5. Let $R_i$ be a $\Gamma_i$-semiring $(1 \leq i \leq n)$. Then, $R_1 \times \cdots \times R_n$ is a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring.

The proof is standard and we hence omit the details. It suffices we define

$$(x_1, \ldots, x_n) \mp (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n),$$

and

$$
\circ : (R_1 \times \cdots \times R_n) \times (\Gamma_1 \times \cdots \times \Gamma_n) \times (R_1 \times \cdots \times R_n) \longrightarrow R_1 \times \cdots \times R_n
$$

by

$$(x_1, \ldots, x_n) \circ (\gamma_1, \ldots, \gamma_n) \circ (y_1, \ldots, y_n) = (x_1\gamma_1 y_1, \ldots, x_n\gamma_n y_n),$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in R_1 \times \cdots \times R_n$ and $(\gamma_1, \ldots, \gamma_n) \in \Gamma_1 \times \cdots \times \Gamma_n$.

In the next lemma, we investigate the behaviour of congruence on the products of $\Gamma$-semirings.

Lemma 5.6. Let $\theta_i$ be a congruence on $(R_i, \Gamma_i)$ for $1 \leq i \leq n$. Then $\theta$ is a congruence on $(R_1 \times \cdots \times R_n, \Gamma_1 \times \cdots \times \Gamma_n)$ where $(a_1, \ldots, a_n)\theta(b_1, \ldots, b_n)$ if and only if $a_i\theta_i b_i$ for all $a_i, b_i \in R_i$ and $1 \leq i \leq n$. 
Proof. If \((x_1, \cdots, x_n) \theta (y_1, \cdots, y_n)\), then \(x_i \theta_i y_i\) for all \(1 \leq i \leq n\). Hence \((x_i + z_i) \theta_i (y_i + z_i)\), for all \(z_i \in R_i\) and \(1 \leq i \leq n\). This implies that
\[
(x_1, \cdots, x_n) \theta (y_1, \cdots, y_n) = (z_1, \cdots, z_n).
\]
Also, \(x_i \theta_i y_i\) for all \(1 \leq i \leq n\) implies that \((x_i \gamma_i z_i) \theta_i (y_i \gamma_i z_i)\) for all \(z_i \in R_i\), \(\gamma_i \in \Gamma_i\) and \(1 \leq i \leq n\). Hence
\[
(x_1, \cdots, x_n) \circ \gamma_i, \cdots, \gamma_n \circ (y_1, \cdots, y_n) \circ \gamma_i, \cdots, \gamma_n \circ (z_1, \cdots, z_n).
\]
Similarly, we can prove that
\[
(z_1, \cdots, z_n) \circ \gamma_i, \cdots, \gamma_n \circ (x_1, \cdots, x_n) \theta (z_1, \cdots, z_n) \circ \gamma_i, \cdots, \gamma_n \circ (y_1, \cdots, y_n).
\]
Therefore, \(\theta\) is a congruence on \((R_1 \times \cdots \times R_n, \Gamma_1 \times \cdots \times \Gamma_n)\). \(\square\)

6 Homomorphism theorems and isomorphism theorems of a \(\Gamma\)-semiring

In the following theorem, we prove an isomorphism theorem of products of \(\Gamma\)-semirings.

Theorem 6.1. Let \(\theta_i\) be a congruence on \((R_i, \Gamma_i)\) for \(1 \leq i \leq n\) and \(\theta\) the congruence on \((R_1 \times \cdots \times R_n, \Gamma_1 \times \cdots \times \Gamma_n)\) defined in Lemma 5.6. Then

\[
\left((R_1 : \theta_1) \times \cdots \times (R_n : \theta_n), \Gamma_1 \times \cdots \times \Gamma_n\right) \cong (R_1 \times \cdots \times R_n : \theta, \Gamma_1 \times \cdots \times \Gamma_n).
\]

Proof. By Theorem 4.5 and Lemmas 5.5 and 5.6, \((R_1 : \theta_1) \times \cdots \times (R_n : \theta_n)\) and \(R_1 \times \cdots \times R_n : \theta\) are \(\Gamma_1 \times \cdots \times \Gamma_n\)-semirings. Define

\[
\psi : (R_1 : \theta_1) \times \cdots \times (R_n : \theta_n) \longrightarrow R_1 \times \cdots \times R_n : \theta
\]

by:

\[
\psi\left(\theta_1(x_1), \cdots, \theta_n(x_n)\right) = \theta(x_1, \cdots, x_n),
\]
for all $x_i \in R_i$ ($1 \leq i \leq n$). We can show that $(\psi, 1_{\Gamma_1 \times \cdots \times \Gamma_n})$ is an isomorphism between

$$\left( (R_1 : \theta_1) \times \cdots \times (R_n : \theta_n), \Gamma_1 \times \cdots \times \Gamma_n \right)$$

and

$$(R_1 \times \cdots \times R_n : \theta, \Gamma_1 \times \cdots \times \Gamma_n).$$

We have

$$\left( \theta_1(x_1), \ldots, \theta_n(x_n) \right) = \left( \theta_1(y_1), \ldots, \theta_n(y_n) \right)$$

$$\iff \theta_i(x_i) = \theta_i(y_i), \ \forall 1 \leq i \leq n$$

$$\iff x_i \theta_i y_i, \ \forall 1 \leq i \leq n$$

$$\iff (x_1 \cdots x_n) \theta(y_1 \cdots y_n)$$

$$\iff \theta(x_1 \cdots x_n) = \theta(y_1 \cdots y_n)$$

$$\iff \psi \left( \theta_1(x_1), \ldots, \theta_n(x_n) \right) = \psi \left( \theta_1(y_1), \ldots, \theta_n(y_n) \right).$$

Hence, $(\psi, 1_{\Gamma_1 \times \cdots \times \Gamma_n})$ is well-defined and one to one. Clearly, $(\psi, 1_{\Gamma_1 \times \cdots \times \Gamma_n})$ is on to. Now, we prove that $(\psi, 1_{\Gamma_1 \times \cdots \times \Gamma_n})$ is a homomorphism. We have

$$\psi \left( \theta_1(x_1), \ldots, \theta_n(x_n) \right) \odot \phi \left( \theta_1(y_1), \ldots, \theta_n(y_n) \right) =$$

$$\psi \left( \theta_1(x_1) \odot \theta_1(y_1), \ldots, \theta_n(x_n) \odot \theta_n(y_n) \right) =$$

$$\psi \left( \theta_1(x_1 + y_1), \ldots, \theta_n(x_n + y_n) \right) =$$

$$\theta \left( x_1 + y_1, \ldots, x_n + y_n \right) =$$

$$\theta(x_1 \cdots x_n) \odot \theta(y_1 \cdots y_n) =$$

$$\psi \left( \theta_1(x_1), \ldots, \theta_n(x_n) \right) \odot \psi \left( \theta_1(y_1), \ldots, \theta_n(y_n) \right).$$

Also, we have

$$\psi \left( \theta_1(x_1), \ldots, \theta_n(x_n) \right) \odot (\gamma_1, \ldots, \gamma_n) \odot \left( \theta_1(y_1), \ldots, \theta_n(y_n) \right) =$$

$$\psi \left( \theta_1(x_1) \odot \gamma_1 \odot \theta_1(y_1), \ldots, \theta_n(x_n) \odot \gamma_n \odot \theta_n(y_n) \right) =$$

$$\psi \left( \theta_1(x_1 \gamma_1 y_1), \ldots, \theta_n(x_n \gamma_n y_n) \right) =$$

$$\theta(x_1 \gamma_1 y_1, \ldots, x_n \gamma_n y_n) =$$

$$\theta(x_1, \ldots, x_n) \odot (\gamma_1, \ldots, \gamma_n) \odot \theta(y_1, \ldots, y_n) =$$
\[ \psi\left(\theta_1(x_1), \cdots, \theta_n(x_n)\right) \odot 1_{\Gamma_1 \times \cdots \times \Gamma_n}(\gamma_1, \cdots, \gamma_n) \odot \psi\left(\theta_1(y_1), \cdots, \theta_n(y_n)\right). \]

Therefore, \((\psi, 1_{\Gamma_1 \times \cdots \times \Gamma_n})\) is an isomorphism. \(\square\)

In the next theorems, we consider the congruence on the \(\Gamma\)-semirings \(R\) induced by the homomorphisms and investigate the corresponding results and properties associated with this congruence on \(R\).

**Theorem 6.2.** Let \((\varphi, g) : (R_1, \Gamma_1) \longrightarrow (R_2, \Gamma_2)\) be a homomorphism. Define the relation \(\theta_{(\varphi, g)}\) on \((R_1, \Gamma_1)\) as follows:

\[ x \theta_{(\varphi, g)} y \iff \varphi(x) = \varphi(y). \]

Then \(\theta_{(\varphi, g)}\) is a congruence on \((R_1, \Gamma_1)\).

**Proof.** Clearly, \(\theta_{(\varphi, g)}\) is an equivalence relation. Suppose that \(x \theta_{(\varphi, g)} y\). We have

\[ \varphi(x) = \varphi(y) \implies \varphi(x) + \varphi(z) = \varphi(y) + \varphi(z) \implies \varphi(x + z) = \varphi(y + z) \]

for all \(z \in R_1\). Thus \((x + z) \theta_{(\varphi, g)} (y + z)\). Also, we have

\[ \varphi(x) = \varphi(y) \implies \varphi(x)g(\gamma)\varphi(z) = \varphi(y)g(\gamma)\varphi(z) \implies \varphi(x\gamma z) = \varphi(y\gamma z) \]

for all \(z \in R_1\) and \(\gamma \in \Gamma_1\). Therefore, \(\theta_{(\varphi, g)}\) is a congruence on \((R_1, \Gamma_1)\). \(\square\)

**Theorem 6.3** Let \((\varphi, g) : (R_1, \Gamma_1) \longrightarrow (R_2, \Gamma_2)\) be a homomorphism. Set \(A = \{I \subseteq R_1 \mid \theta_{(\varphi, g)} \subseteq I \times I\}\) and \(B = \{J \mid J \subseteq R_2\}\). Then, there exists an 1-1 mapping from \(A\) to \(B\).

**Proof.** Define \(\psi : A \longrightarrow B\) by \(\psi(I) = \varphi(I)\). Clearly, \(\psi\) is well-defined. Suppose that \(\psi(I_1) = \psi(I_2)\). Then \(\varphi(I_1) = \varphi(I_2)\). Also we can see that

\[ x \in I_1 \implies \varphi(x) \in \varphi(I_1) = \varphi(I_2) \]
\[ \implies \exists y \in I_2, \varphi(x) = \varphi(y) \]
\[ \implies (x, y) \in \theta_{(\varphi, g)} \subseteq I_2 \times I_2 \]
\[ \implies x \in I_2 \]
\[ \implies I_1 \subseteq I_2. \]
Similarly, $I_2 \subseteq I_1$ and so $I_1 = I_2$ and hence, $\psi$ is one to one. \hfill \square

**Theorem 6.4.** Let $(R_1, \Gamma_1) \xrightarrow{(\varphi_1, g_1)} (R_2, \Gamma_2) \xrightarrow{(\varphi_2, g_2)} (R_3, \Gamma_3)$ be a sequence of homomorphisms. Then

$$(\psi, g) : (R_1 \times R_1, \Gamma_1 \times \Gamma_1) \longrightarrow (R_2 \times R_2, \Gamma_2 \times \Gamma_2),$$

defined by $\psi(x, y) = (\varphi_1(x), \varphi_1(y))$ and $g(\gamma, \beta) = (g_1(\gamma), g_1(\beta))$ for all $x, y \in R_1$ and $\gamma, \beta \in \Gamma_1$, is a homomorphism such that $\psi(\theta_{(\varphi_1, g_1)}) \subseteq \theta_{(\varphi_2, g_2)}$. Moreover, if $(\varphi_1, g_1)$ is surjective and $(\varphi_2, g_2)$ is injective, then we have $\psi(\theta_{(\varphi_1, g_1)}) = \theta_{(\varphi_2, g_2)}$.

**Proof.** It is trivial that $(\psi, g)$ is a homomorphism. Hence, we have

$$\psi(a, b) \in \psi(\theta_{(\varphi_1, g_1)}), \quad (a, b) \in \theta_{(\varphi_1, g_1)} \implies \varphi_1(a) = \varphi_1(b)$$

$$\implies \varphi_2(\varphi_1(a)) = \varphi_2(\varphi_1(b))$$

$$\implies (\varphi_1(a), \varphi_1(b)) \in \theta_{(\varphi_2, g_2)}$$

$$\implies \psi(a, b) \in \theta_{(\varphi_2, g_2)}.$$ 

Thus, $\psi(\theta_{(\varphi_1, g_1)}) \subseteq \theta_{(\varphi_2, g_2)}$. Now, if $(\varphi_1, g_1)$ is surjective and $(\varphi_2, g_2)$ is injective, then we have $\psi(\theta_{(\varphi_1, g_1)}) = \theta_{(\varphi_2, g_2)}$. It suffices to prove that $\theta_{(\varphi_2, g_2)} \subseteq \psi(\theta_{(\varphi_1, g_1)})$. Hence, we have

$$(t, t') \in \theta_{(\varphi_2, g_2)} \implies \varphi_2(t) = \varphi_2(t')$$

$$\implies \exists a, b \in R, \ \varphi_1(a) = t, \ \varphi_1(b) = t'$$

$$\implies (t, t') = \psi(a, b) = (\varphi_1(a), \varphi_1(b))$$

$$\implies (t, t') \in \psi(\theta_{(\varphi_1, g_1)}).$$

This shows that $\theta_{(\varphi_2, g_2)} \subseteq \psi(\theta_{(\varphi_1, g_1)})$, and the proof is completed. \hfill \square

**Theorem 6.5.** Let $(R_1, \Gamma_1) \xrightarrow{(\varphi_1, g_1)} (R_2, \Gamma_2) \xrightarrow{(\varphi_2, g_2)} (R_3, \Gamma_3)$ be a sequence of homomorphisms. Then $\text{Im} \varphi_1 \times \text{Im} \varphi_1 \subseteq \theta_{(\varphi_2, g_2)}$ if and only if $\varphi_2 \circ \varphi_1$ is constant.

**Proof.** The proof of the necessary part is routine and we only prove the sufficiency part.
(⇒): Let \( x, y \in R_1 \). Then, \( (\varphi_1(x), \varphi_1(y)) \in \text{Im} \varphi_1 \times \text{Im} \varphi_1 \subseteq \theta_{(\varphi_2, \triangleright)} \). Hence, \( \varphi_2(\varphi_1(x)) = \varphi_2(\varphi_1(y)) \). This show that \( \varphi_2 \circ \varphi_1 \) is a constant. \( \square \)

Finally, by the congruence on the \( \Gamma \)-semiring induced by homomorphism, we are able to establish some isomorphism theorems and investigate the commutativity of some diagrams.

**Theorem 6.6.** (Isomorphism Theorem) If \( (\varphi, g) : (R_1, \Gamma_1) \longrightarrow (R_2, \Gamma_2) \) is an epimorphism, then there exists an unique isomorphism \( (\psi, g) : (R_1 : \theta_{(\varphi, g)}, \Gamma_1) \longrightarrow (R_2, \Gamma_2) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
(R_1, \Gamma_1) & \xrightarrow{(\varphi, g)} & (R_2, \Gamma_2) \\
\Pi_{R_1} \downarrow & & \downarrow (\psi, g) \\
(R_1 : \theta_{(\varphi, g)}, \Gamma_1) & & \\
\end{array}
\]

where \( \Pi_{R_1} : R_1 \longrightarrow R_1 : \theta_{(\varphi, g)} \) is defined by \( \Pi_{R_1}(x) = \theta_{(\varphi, g)}(x) \) for all \( x \in R_1 \) and \( 1_{\Gamma_1} \) is identity.

**Proof.** Define \( \psi : R_1 : \theta_{(\varphi, g)} \longrightarrow R_2 \) by \( \psi(\theta_{(\varphi, g)}(x)) = \varphi(x) \) for all \( x \in R_1 \). Then we have

\[ \theta_{(\varphi, g)}(x) = \theta_{(\varphi, g)}(y) \iff x \theta_{(\varphi, g)} y \iff \varphi(x) = \varphi(y), \]

and hence \( \psi \) is well defined and is a 1-1 mapping. Now, \( (\psi, g) \) is a homomorphism. We have

\[
\psi\left(\theta_{(\varphi, g)}(x) \oplus \theta_{(\varphi, g)}(y)\right) = \psi\left(\theta_{(\varphi, g)}(x + y)\right) = \varphi(x + y) = \varphi(x) + \varphi(y) = \psi\left(\theta_{(\varphi, g)}(x)\right) + \psi\left(\theta_{(\varphi, g)}(y)\right).
\]

We deduce that

\[
\psi\left(\theta_{(\varphi, g)}(x) \odot \gamma \odot \theta_{(\varphi, g)}(y)\right) = \psi\left(\theta_{(\varphi, g)}(x \gamma y)\right) = \varphi(x \gamma y) = \varphi(x)g(\gamma)\varphi(y) = \psi\left(\theta_{(\varphi, g)}(x)\right)g(\gamma)\psi\left(\theta_{(\varphi, g)}(y)\right).
\]
Therefore, \((\psi, g)\) is a homomorphism. Also \(\varphi(x) = \psi(\theta_{(\varphi, g)}(x)) = \psi(\Pi R_1(x))\)
and \(g \circ 1_{\Gamma_1} = g\) which imply that the diagram is commutative. Let

\[
(\tilde{\psi}, \tilde{g}) : (R_1 : \theta_{(\varphi, g)}, \Gamma_1) \longrightarrow (R_2, \Gamma_2)
\]

be such that \(\tilde{\psi} \circ \Pi R_1 = \varphi\). Then, We have

\[
\tilde{\psi}(\theta_{(\varphi, g)}(x)) = \tilde{\psi}(\Pi R_1(x)) = \varphi(x) = \psi(\Pi R_1(x)) = \psi(\theta_{(\varphi, g)}(x)).
\]

Thus, \((\psi, g)\) is unique and the proof is completed. \(\square\)

**Theorem 6.7.** Let \((R_1, \Gamma_1) \xrightarrow{\varphi_1, g_1} (R_2, \Gamma_2) \xrightarrow{\varphi_2, g_2} (R_3, \Gamma_3)\) be a sequence of homomorphisms. Then, there exists an unique homomorphism\((\psi, g_1) : (R_1 : \theta_{(\varphi_1, g_1)}, \Gamma_1) \longrightarrow (R_2 : \theta_{(\varphi_2, g_2)}, \Gamma_2)\)
such that the following diagram is commutative:

\[
\begin{array}{ccc}
(R_1, \Gamma_1) & \xrightarrow{(\varphi_1, g_1)} & (R_2, \Gamma_2) \\
\downarrow & & \downarrow \\
(R_1 : \theta_{(\varphi_1, g_1)}, \Gamma_1) & \xrightarrow{(\psi, g_1)} & (R_2 : \theta_{(\varphi_2, g_2)}, \Gamma_2)
\end{array}
\]

Moreover, if \((\varphi_1, g_1)\) is onto and \((\varphi_2, g_2)\) is 1-1, then \((\psi, g_1)\) is an isomorphism.

**Proof.** Define \(\psi : R_1 : \theta_{(\varphi_1, g_1)} \longrightarrow R_2 : \theta_{(\varphi_2, g_2)}\) by \(\psi(\theta_{(\varphi_1, g_1)}(x)) = \theta_{(\varphi_2, g_2)}(\varphi_1(x))\). Then, \(\psi\) is well-defined.

Finally, we need prove that \((\psi, g_1)\) is a homomorphism. We have

\[
\psi(\theta_{(\varphi_1, g_1)}(x) \oplus \theta_{(\varphi_1, g_1)}(y)) = \psi(\theta_{(\varphi_1, g_1)}(x + y)) = \psi(\theta_{(\varphi_2, g_2)}(\varphi_1(x) + \varphi_1(y)) = \theta_{(\varphi_2, g_2)}(\varphi_1(x) \oplus \varphi_1(y)) = \psi(\theta_{(\varphi_1, g_1)}(x)) \oplus \psi(\theta_{(\varphi_1, g_1)}(y)).
\]

Also, we have
\[
\psi \left( \theta_{(\varphi_1, g_1)}(x) \circ \gamma \circ \theta_{(\varphi_1, g_1)}(y) \right) = \psi \left( \theta_{(\varphi_1, g_1)}(x \gamma y) \right) \\
= \theta_{(\varphi_2, g_2)}(\varphi_1(x \gamma y)) \\
= \theta_{(\varphi_2, g_2)}(\varphi_1(x) g_1(\gamma) \varphi_1(y)) \\
= \theta_{(\varphi_2, g_2)}(\varphi_1(x)) \circ g_1(\gamma) \circ \theta_{(\varphi_2, g_2)}(\varphi_1(y)) \\
= \psi \left( \theta_{(\varphi_1, g_1)}(x) \circ g_1(\gamma) \circ \psi \left( \theta_{(\varphi_1, g_1)}(y) \right) \right).
\]

Therefore, \((\psi, g_1)\) is a homomorphism. Also, we have

\[
\psi \left( \Pi_{R_1}(x) \right) = \psi \left( \theta_{(\varphi_1, g_1)}(x) \right) = \theta_{(\varphi_2, g_2)}(\varphi_1(x)) = \Pi_{R_2}(\varphi_1(x)),
\]

and \(g_1 \circ 1_{\Gamma_1} = 1_{\Gamma_2} \circ g_1\). This shows that the diagram is commutative.

Let \((\tilde{\psi}, g_1) : (R_1 : \theta_{(\varphi_1, g_1)}, \Gamma_1) \longrightarrow (R_2 : \theta_{(\varphi_2, g_2)}, \Gamma_2)\) be a homomorphism which makes the diagram commutative. Then, we have

\[
\tilde{\psi} \left( \theta_{(\varphi_1, g_1)}(x) \right) = \tilde{\psi} \left( \Pi_{R_1}(x) \right) = \Pi_{R_2}(\varphi_1(x)) = \theta_{(\varphi_2, g_2)}(\varphi_1(x)) = \psi \left( \theta_{(\varphi_1, g_1)}(x) \right).
\]

Thus, \((\psi, g_1)\) is unique and the proof is completed. \(\square\)

References


Ideals, congruences, homomorphisms and isomorphisms


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