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Generalized Functions for Double Sumudu Transformation

S. K. Q. Al-Omari

Department of Applied Sciences
Faculty of Engineering Technology
Al-Balqa Applied University
Amman 11134, Jordan
s.k.q.alomari@fet.edu.jo

Abstract. For a probably first time investigation the generalized double Sumudu transform of a function of two variables is extended and many of its distributional properties are obtained in the sense of the space of distributions of bounded support. Further, we establish a generalization of the double Sumudu transformation for a space of integrable Boehmians.

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1. Introduction

Several integral transforms were extensively used and applied to solve differential equations, such as: Laplace, Fourier, Hankel and convolution transforms, to name but a few (See $[1, 2, 3, 4, 5, 6]$, [10] and [11]).

Weerakoon in [13] has discussed the Sumudu transform of one variable of partial differential equations, whereas, Belgacem et al., in [7] , shows that the Sumudu transform has deeper connections with the Laplace transform. However, the approach here is somewhat different and interesting. This paper aims at extending the double Sumudu transform to a space of distributions of bounded support and establishes certain theorems and generalizations of results from [9] . Such an extension propably opens an avenue of the double transform of generalized functions, such as: distributions, ultradistributions, tempered ultradistributions and possible Boehmian spaces in a way similar to that of the rest of integral transforms.

The double Sumudu transformation of a function $f(t,x)$,$t,x>0$,of two variables is defined by [9]

$$\mathbf{S}(u,v)\cong\mathbf{S}(f(t,x))(u,v)\cong\int_0^\infty\int_0^\infty f(t,x)\frac{1}{uv}\exp\left(-\frac{t}{u}-\frac{x}{v}\right)dtdx, \tag{1.1}$$

provided the integral exists and $f(t,x)$ is a function which can be expressed as a convergent series.

Let f,g be integrable functions of two variables then in reference to [9, Theorem 1] the double convolution of f and g is given by

$$\left(f*^2g\right)(x,y)=\int_0^\infty\int_0^\infty f(x-\tau_1,y-\tau_2)\,g(\tau_1,\tau_2)\,d\tau_1d\tau_2, \tag{1.2}$$

where $*^2$ is a notation of the double convolution with respect to x and y .

2. Double Sumudu Transform of Bounded Support

Let $\mathbf{K}$ be a compact subset of $I\,(I=(0,\infty))$. By $\mathbf{E}(I\times I)$ we denote the test function space of all complex-valued infinitely smooth functions on $I\times I$ (with arbitrary support) such that

$$\sup_{(t,x)\in\mathbf{K}\times\mathbf{K}}\left|\frac{\partial^{k+m}}{\partial t^k\partial x^m}\phi(t,x)\right| \tag{2.1}$$

is finite for all non-negative integers k and m.

A sequence $\left(\phi_j\right)$ is said to converge in the sense of of the topology of $\mathbf{E}(I\times I)$ if and only if for every nonnegative and fixed integers k,m, the sequence $\left(\frac{\partial^{k+m}}{\partial t^k\partial x^m}\phi_j(t,x)\right)$ converges uniformly on every compact subset of $I\times I$. The space $\mathbf{E}(I\times I)$ is complete and the strong dual $\mathbf{E}'(I\times I)$ of $\mathbf{E}(I\times I)$ consists of distributions of compact support .

As the definition (1.1) shows, the kernel function $e^{-(t/u+x/v)}/(uv)$ of the Sumudu transform is a smooth function satisfying

$$\sup_{(t,x)\in\mathbf{K}\times\mathbf{K}}\left|\frac{\partial^{k+m}}{\partial t^k\partial x^m}\left(e^{-(t/u+x/v)}/(uv)\right)\right|<\infty \tag{2.2}$$

when $\mathbf{K}\times\mathbf{K}$ is allowed to range over compact subsets of $I\times I$ and for all positive real numbers u and v.

Hence, for every $f\in\mathbf{E}'(I\times I)$, (3.2) leads us to define the distributional double Sumudu transform of the distribution f of bounded support as

$$\hat{\mathbf{S}}(u,v)\cong\hat{\mathbf{S}}(f(t,x))(u,v)\cong\left\langle f(t,x),e^{-(t/u+x/v)}/(uv)\right\rangle, \tag{2.3}$$

for arbitrary real numbers $u,v,(u,v)\in I\times I$.

Theorem 2.1 *The distributional double Sumudu transform* $\hat{\mathbf{S}}$ *is linear .*

Proof of this theorem is straightforward consequence of the property of dilation of distributions. Thus, details are avoided.

Theorem 2.2 *Lef f be a distribution in $\mathbf{E}'(I \times I)$ and let g be defined as*

$$g(t,x) = \left\{ \begin{array}{c} f(t-\tau_1, x-\tau_2), t \geq \tau_1, x \geq \tau_2 \\ 0, t < \tau_1, x < \tau_2 \end{array} \right\},$$

then $\hat{\mathbf{S}}_2(u,v) = e^{-(\tau_1/u+\tau_2/u)}\hat{\mathbf{S}}_1(u,v)$, where $\hat{\mathbf{S}}_2$ and $\hat{\mathbf{S}}_1$ are the distributional double Sumudu transforms of f and g, respectively.

Proof It is clear that $g \in \mathbf{E}'(I \times I)$. Therefore, the translation property of distributions through τ $[14, \text{p.}26]$, when extended to τ_1 and τ_2, implies

$$\hat{\mathbf{S}}_2(u,v) = \left\langle f(t-\tau_1, x-\tau_2), e^{-(t/u+x/v)}/uv \right\rangle = e^{-(\tau_1/u+\tau_2/v)}\hat{\mathbf{S}}_1(u,v).$$

Hence, the theorem.

Theorem 2.3 *Let $f \in \mathbf{E}'(I \times I)$ and $\hat{\mathbf{S}}(u,v)$ be the double Sumudu transform of f, then $\frac{\partial^k}{\partial u^k}\hat{\mathbf{S}}(u,v) = \left\langle f(t,x), \frac{\partial^k}{\partial u^k}\left(e^{-(t/u+x/v)}/uv\right)\right\rangle$*,and

$$\frac{\partial^m}{\partial v^m}\hat{\mathbf{S}}(u,v) = \left\langle f(t,x), \frac{\partial^m}{\partial v^m}\left(e^{-(t/u+x/v)}/uv\right)\right\rangle, \tag{2.4}$$

where $\frac{\partial^k}{\partial u^k}, \frac{\partial^m}{\partial v^m}$ stands for the respective k-th and m-th derivatives with respect to u and v.

Theorem 2.4. *Let $f \in \mathbf{E}'(I \times I)$ and $\hat{\mathbf{S}}(u,v)$ be the distributional double Sumudu transform of f, then*

$$(i)\ \hat{\mathbf{S}}\left(t^n \frac{\partial^n}{\partial t^n} f(t,x)\right)(u,v) = u^n \frac{\partial^n}{\partial u^n}\hat{\mathbf{S}}(u,v);$$

$$(ii)\ \hat{\mathbf{S}}\left(x^n \frac{\partial^n}{\partial x^n} f(t,x)\right)(u,v) = v^n \frac{\partial^n}{\partial v^n}\hat{\mathbf{S}}(u,v);$$

$$(iii)\ \hat{\mathbf{S}}\left(t^n x^m \frac{\partial^{n+m}}{\partial t^n \partial x^m} f(t,x)\right)(u,v) = u^n v^m \frac{\partial^{n+m}}{\partial u \partial v^n}\hat{\mathbf{S}}(u,v).$$

Proof We prove the first part of the theorem. Proofs of the second and third parts are analoguous. In consideration of the properties of double Sumudu transform and (2.3), we get

$$\frac{\partial^n}{\partial u^n}\hat{\mathbf{S}}(u,v) = \frac{\partial^n}{\partial u^n}\left\langle f(t,x), e^{-(t/u+x/v)}/uv\right\rangle = \frac{\partial^n}{\partial u^n}\left\langle f(tu,tv), e^{-(t+x)}\right\rangle.$$

i.e. $\frac{\partial^n}{\partial u^n}\hat{\mathbf{S}}(u,v) = \frac{\partial^n}{\partial u^n}\left\langle f(tu,tx), e^{-(t+x)}\right\rangle = \left\langle t^n \frac{\partial^n}{\partial t^n} f(tu,tx), e^{-(t+x)}\right\rangle$. i.e. $\frac{\partial^n}{\partial u^n}\hat{\mathbf{S}}(u,v) = \frac{1}{u^n}\hat{\mathbf{S}}\left(t^n \frac{\partial^n}{\partial t^n} f(t,x)\right)(u,v)$. This complete the proof of the theorem.

Theorem 2.5. *Let $f \in \mathbf{E}'(I \times I)$ and $\hat{\mathbf{S}}(u,v)$ be its corresponding distributional double Sumudu transform, then*

$$\hat{\mathbf{S}}\left(e^{at+bx} f(t,x)\right)(u,v) = (1/(1-au)(1-bu))\,\hat{\mathbf{S}}(uv/(1-au)(1-bu)).$$

Proof of this theorem results from Equation 1.4 and the basic properties of differentiation.

Theorem 2.6. *Let* $f \in \mathbf{E}'(I \times I)$ *and* $\hat{\mathbf{S}}(u,v)$ *be the distributional double Sumudu transform of* f, *then*

$$\hat{\mathbf{S}}(f(at,bx))(u,v) = \hat{\mathbf{S}}(f(t,x))(au,bv).$$

Proof. Applying the property of change under scale of Sumudu transforms, our theorem can be easily established.

Following is a theorem, which deals with multiplication of a distribution $f(t,x)$ by a positive power of t and x.

Theorem 2.7. *Let* $f \in \mathbf{E}'(I \times I)$ *and* $\hat{\mathbf{S}}(u,v)$ be *the distributional Sumudu transform of* f, *then*

$(i)\, \hat{\mathbf{S}}(tf(t,x))(u,v) = u^2 \frac{\partial}{\partial u}\hat{\mathbf{S}}(f(t,x))(u,v) + u\hat{\mathbf{S}}(f(t,x))(u,v).$

$(ii)\, \hat{\mathbf{S}}(xf(t,x))(u,v) = v^2 \frac{\partial}{\partial v}\hat{\mathbf{S}}(f(t,x))(u,v) + v\hat{\mathbf{S}}(f(t,x))(u,v).$

Proof We prove Part (i) of the theorem since the second part is similar . Let $f \in \mathbf{E}'(I \times I)$ then employing (2.3) and Theorem 2.3 we have

$$\frac{\partial}{\partial u}\hat{\mathbf{S}}(u,v) = \left\langle f(t,x), \frac{\partial}{\partial u}\left(e^{-(t/u+x/v)}/uv\right)\right\rangle.$$

With the aid of the rules of differentiation, simple calculations yield

$\frac{\partial}{\partial u}\hat{\mathbf{S}}(u,v) = (1/u^2)\left\langle tf, e^{-(t/u+x/v)}/uv\right\rangle - (1/u)\left\langle f, e^{-(t/u+x/v)}/uv\right\rangle.$

Thus $\hat{\mathbf{S}}(tf(t,x))(u,v) = u^2 \frac{\partial}{\partial u}\hat{\mathbf{S}}(u,v) + u\hat{\mathbf{S}}(u,v).$

Fortunately, we may proceed as in (i) to derive Part (ii) of the theorem. Detailed proof is avoided.

Moreover Theorem 2.7 can be easily extended to multiplication by t^n and $x^n, n \in \mathbb{N}$. The desired proof of the extended result can, then, be automatically constructed with the help of the principle of mathmatical induction on n .

Theorem 2.8. *Let* $\hat{\mathbf{S}}(f(t,x))(u,v)$ *be the double distributional Sumudu transform of* $f(t,x)$, *then the following holds*

$(a)\, \hat{\mathbf{S}}(txf)(u,v) = u^2v^2\frac{\partial^2\hat{\mathbf{S}}(u,v)}{\partial u\partial v} + v\hat{\mathbf{S}}(tf)(u,v) + u\hat{\mathbf{S}}(xf)(u,v) - uv\hat{\mathbf{S}}(f)(u,v),$

$(b)\, \hat{\mathbf{S}}(t^2f)(u,v) = u^4\frac{\partial^2\hat{\mathbf{S}}(u,v)}{\delta u^2} + 4u\hat{\mathbf{S}}(tf)(u,v) - 2u^2\hat{\mathbf{S}}(f)(u,v),$

$(c)\, \hat{\mathbf{S}}(x^2f)(u,v) = v^4\frac{\partial^2\hat{\mathbf{S}}(u,v)}{\partial v^2} + 4v\hat{\mathbf{S}}(xf)(u,v) - 2v^2\hat{\mathbf{S}}(f)(u,v).$

Proof of (a) Let $f(t,x) \in \mathbf{E}'(I \times I)$ then employing (2.3) and theorem (2.3) we have

$$\frac{\partial^2\hat{\mathbf{S}}(u,v)}{\partial u\partial v} = \left\langle f(t,x), \frac{\partial}{\partial u}\frac{e^{-t/u}}{u}\frac{\partial}{\partial v}\frac{e^{-x/v}}{v}\right\rangle.$$

Rules of differentiations and simple calculations yield

$$\frac{\partial^2}{\partial u\partial v}\hat{\mathbf{S}}(u,v) = \frac{1}{u^2v^2}\left\langle xtf, \frac{e^{-(t/u+x/v)}}{uv}\right\rangle - \frac{1}{u^2v}\left\langle tf, \frac{e^{-(t/u+x/v)}}{uv}\right\rangle$$
$$-\frac{1}{uv^2}\left\langle xf, \frac{e^{-(t/u+x/v)}}{uv}\right\rangle + \frac{1}{uv}\left\langle f, \frac{e^{-(t/u+x/v)}}{uv}\right\rangle.$$

Upon muttiplying both side by u^2v^2 and considering the definition we get i.e. $\hat{\mathbf{S}}(txf(t,x))(u,v) = u^2v^2\frac{\partial^2\hat{\mathbf{S}}(u,v)}{\partial u\partial v} + v\hat{\mathbf{S}}(tf)(u,v) + u\hat{\mathbf{S}}(xf)(u,v) - uv\hat{\mathbf{S}}(f)(u,v).$

Now, to prove the second part of the theorem we use the definition of the distributional Sumudu transform and its second partial derivative with respect to u as follows.

$\frac{\partial^2 \hat{\mathbf{S}}(u,v)}{\partial u^2} = \frac{1}{u^4}\left\langle t^2 f, \frac{e^{-(t/u+x/v)}}{uv}\right\rangle - \frac{4}{u^3}\left\langle tf, \frac{e^{-(t/u+x/v)}}{uv}\right\rangle + \frac{1}{u^2}\left\langle 2f, \frac{e^{-(t/u+x/v)}}{uv}\right\rangle.$

Upon multiplying both sides by u^4 and rearranging we have

$u^4 \frac{\partial^2 (u,v)}{\partial u^2} = \hat{\mathbf{S}}\left(t^2 f\right)(u,v) - 4u\hat{\mathbf{S}}\left(tf\right)(u,v) + 2u^2\hat{\mathbf{S}}\left(f\right)(u,v)$

or $\hat{\mathbf{S}}\left(t^2 f\right)(u,v) = u^4 \frac{\partial^2 \hat{\mathbf{S}}(u,v)}{\partial u^2} + 4u\hat{\mathbf{S}}\left(tf\right)(u,v) - 2u^2\hat{\mathbf{S}}\left(f\right)(u,v)$

Proof of Part(c) is similar to that of Part(b). This completes the proof of the theorem.

Corrollary 2.9 *Let* $\hat{\mathbf{S}}$ *be the distributional double Sumudu transform of* f *then*

$(i)\, \hat{\mathbf{S}}\left(x^2 f(t,x)\right)(u,v) = u^4 \frac{\partial^2 \hat{\mathbf{S}}(u,v)}{\partial u^2} + 4u^3 \frac{\partial \hat{\mathbf{S}}(u,v)}{\partial u} + 2u^2 \hat{\mathbf{S}}(u,v),$

$(ii)\, \hat{\mathbf{S}}\left(t^2 f(t,x)\right)(u,v) = t^4 \frac{\partial^2 \hat{\mathbf{S}}(u,v)}{\partial v^2} + 4v^2 \frac{\partial \hat{\mathbf{S}}(u,v)}{\partial v} + 2v^2 \hat{\mathbf{S}}(u,v).$

Proof is a straightforword consequence of Theorem 2.7 and Theorem 2.8 by invoking Part(a) and (b) of Theorem 2.7 into Part (b) and (c) of Theorem 2.8, respectively.

3. Convolution of Double Sumudu Transform

Let f and g be distributions of compact support in $\mathbf{E}'(I \times I)$, then we define a convolution between f and g in a natural way as

$$\left\langle \left(f *^2 g\right)(t,x), \phi(t,x)\right\rangle = \left\langle f(t,x), \left\langle g(\tau_1,\tau_2), \phi(t+\tau_1, x+\tau_2)\right\rangle\right\rangle, \tag{3.1}$$

where $\phi \in \mathbf{E}(I \times I)$.

The above definition is meaningful provided

$$\psi(t,x) = \left\langle g(\tau_1,\tau_2), \phi(t+\tau_1, x+\tau_2)\right\rangle \in \mathbf{E}(I \times I).$$

Theorem 3.1. *Let* $g \in \mathbf{E}'(I \times I)$ *and* $\phi \in \mathbf{E}(I \times I)$.
If $\psi(t,x) = \left\langle g(\tau_1,\tau_2), \phi(t+\tau_1, x+\tau_2)\right\rangle$*,then* ψ *is infinitely differentiable and*

$$\frac{\partial^{k+m}}{\partial t^k \partial x^m}\psi(t,x) = \left\langle g(\tau_1,\tau_2), \frac{\partial^{k+m}}{\partial t^k \partial x^m}\phi(t+\tau_1, x+\tau_2)\right\rangle, k,m = 1,2,....$$

Proof of the above desired result can be established by an argument which is alike to that obtained for Theorem 4.5.1 from [11, p.p.130] and, thus, we avoid the same.

Theorem 3.2.(The Convolution Theorem) *If* $\hat{\mathbf{S}}_1(u,v)$ *and* $\hat{\mathbf{S}}_2(u,v)$ *are the distributional Sumudu transforms of* f *and* g*, respectively, then*

$$\hat{\mathbf{S}}\left(\left(f *^2 g\right)(t,x)\right)(u,v) = uv\hat{\mathbf{S}}_1(u,v)\,\hat{\mathbf{S}}_2(u,v).$$

Proof. Employing the translation property and Equation 3.1, we get
$\hat{\mathbf{S}}\left(\left(f *^2 g\right)(t,x)\right)(u,v) = \left\langle \left(f *^2 g\right)(t,x), e^{-(t/u+x/v)}/uv\right\rangle$. Thus, the theorem is completely proved.

Theorem 3.3. *Let* $f, g \in \mathbf{E}'(I \times I)$ *and* $\hat{\mathbf{S}}(g)(u,v), \hat{\mathbf{S}}(f)(u,v)$ *be their respective distributional double Sumudu transforms, then*

$(i)\, \hat{\mathbf{S}}\left(\frac{\partial^m}{\partial x^m}(f *^2 g)(t,x)\right)(u,v) = uv\hat{\mathbf{S}}\left(\frac{\partial^m}{\partial t^m} f\right)(u,v)\,\hat{\mathbf{S}}(g)(u,v),$

$(ii)\, \hat{\mathbf{S}}\left(\frac{\partial^m}{\partial t^m}(f *^2 g)(t,x)\right)(u,v) = uv\hat{\mathbf{S}}(f)(u,v)\,\hat{\mathbf{S}}\left(\frac{\partial^m}{\partial t^m} g\right)(u,v).$

Proof We intend to prove Part (i) of the theorem since proof of the second part is similar. With the aid of the fact $\mathcal{D}^m(f * g) = f^{(m)} * g = g * f^{(m)}$,which can be extended to the double sense $\frac{\partial^m}{\partial t^m}(f *^2 g)(t,x) = \frac{\partial^m f}{\partial t^m} *^2 g = g *^2 \frac{\partial^m f}{\partial t^m}$

4. Integrable Boehmians for Double Sumudu Transform

Let L^1 be the space of all Lebesgue integrable functions on the positive real line, then, as in [10] , a sequence $(\delta_n)_{n=1}^{\infty}$ of continuous real functions over $I \times I$ is called a delta sequence if and only if

$(i) \int\limits_0^\infty \int\limits_0^\infty \delta_n(t,x) = 1;$

$(ii) \int\limits_0^\infty \int\limits_0^\infty |\delta_n| < M,$ *for some positive* $M \in \mathbb{R}$ *and all* $n \in \mathbb{N}$;

$(iii) \int\limits_{|(t,x)|>\varepsilon} |\delta_n(t,x)|\, dtdx \to 0$ *as* $n \to \infty$ *for every* $\varepsilon > 0$.

The space of all integrable Boehmians is denoted by $\mathbf{B}_{\mathbf{L}^1 \times \mathbf{L}^1}$, which is a convolution algebra with the following operations $\lambda[f_n/\delta_n] = [\lambda f_n/\delta_n]$, $[f_n/\delta_n] + [g_n/\xi_n] = [(f_n *^2 \xi_n + g_n *^2 \delta_n)/\delta_n *^2 \xi_n]$, and $[f_n/\delta_n] *^2 [g_n/\xi_n] = [f_n *^2 g_n/\delta_n *^2 \xi_n]$ (see [10]) .

Lemma 4.1 *If* $[f_n/\delta_n] \in \mathbf{B}_{\mathbf{L}^1 \times \mathbf{L}^1}$, *then the sequence*

$$\mathbf{S}(f_n(t,x))(u,v) = \int\limits_0^\infty \int\limits_0^\infty \tfrac{1}{uv} \exp\left(-\tfrac{t}{u} - \tfrac{x}{v}\right) f_n(t,x)\, dtdx$$

converges uniformly on each compact set $\mathbf{K} \times \mathbf{K}$ *in* $I \times I$.

Proof If (δ_n) is a sequence, then its double Sumudu transforms $\left(\mathbf{S}(\delta_n) = \tilde{\delta}_n\right)$ converges uniformly on each compact subset to the function $\frac{1}{uv}$. Hence, for each $\mathbf{K} \times \mathbf{K}, \tilde{\delta}_n$ is positive on $\mathbf{K} \times \mathbf{K}$ for almost $k \in \mathbb{N}$ and

$$\mathbf{S}(f_n(t,x))(u,v) = \mathbf{S}f_n(u,v)\tfrac{\tilde{\delta}_k}{\tilde{\delta}_k} = \tfrac{\mathbf{S}f_k}{\tilde{\delta}_k}\tilde{\delta}_n \text{ on } \mathbf{K} \times \mathbf{K}.$$

i.e. $\mathbf{S}f_n \to \frac{\mathbf{S}f_k}{uv\tilde{\delta}_k}$ as $n \to \infty$ on $\mathbf{K} \times \mathbf{K}\left(\tilde{\boldsymbol{\delta}}(u,v) \to \frac{1}{uv} \text{ as } n \to \infty\right)$.

Based on this result, we define the Double Sumudu transform of an integrable Boehmian as

$$\Psi[f_n/\phi_n] = \lim\nolimits_{n\to\infty} f_n, \tag{4.1}$$

where the limit ranges over compact subsets of $I \times I$. Thus, the Sumudu transform of an integrable Boehmian is a continuous function.

As a next step, we claim the concept in (4.1) is well-defined. For, let $\beta_1 = [f_n/\phi_n]$ and $\beta_2 = [g_n/\psi_n]$ be in $\mathbf{B}_{\mathbf{L}^1\times\mathbf{L}^1}$ such that $\beta_1 = \beta_2$. Then, $[f_n/\phi_n] = [g_n/\psi_n]$ implies $f_n *^2 \psi_m = g_m *^2 \phi_n, m, n \in \mathbb{N}$. Therefore, $\mathbf{S}(f_n *^2 \psi_m) = \mathbf{S}(g_m *^2 \phi_n) = \mathbf{S}(g_n *^2 \phi_m)$.

From Theorem 3.2 and (4.1) we have $\lim_{n\to\infty} \mathbf{S}f_n = \lim_{n\to\infty} \mathbf{S}g_n$, over compact subsets of $(0,\infty)$. i.e. $\Psi[f_n/\phi_n] = \Psi[g_n/\phi_n]$.

Theorem 4.2 *Let* $F, G \in \mathbf{B}_{\mathbf{L}^1\times\mathbf{L}^1}$, *then* $(i)\ \Psi(\lambda F) = \lambda\Psi F$ *and* $\Psi(F+G) = \Psi F + \Psi G$, $(ii)\ \Psi(F *^2 G) = uv\Psi F\Psi G$, $(iii)\ \Psi\left(\frac{\partial^{k+m}}{\partial t^k \partial x^m} F\right) = \frac{(-1)^{k+m}}{u^k v^m}\Psi F$, (v)*If* $\Psi F = 0$ *then* $F = 0$, (vi) *If* $\Delta - \lim_{n\to\infty} F_n = F$ *then* $\Psi F_n \to \Psi F$ *unifomly on each compact set.*

Proof Properties $(i)-(v)$ can directly be established from the corresponding properties of the Sumudu transform. Part (vi) can be proved in a manner similar to that of [10, Theorem 2, Part(f)]. The theorem is, thus, completely proved.

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