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Subgroups Lattice of Symmetric Group S_4

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Abstract

In this paper, we determine all of subgroups of symmetric group S_4 by applying Lagrange theorem and Sylow theorem. First, we observe the multiplication table of S_4, then we determine all possibilities of every subgroup of order n, with n is the factor of order S_4. We found 30 subgroups of S_4. The diagram of lattice subgroups of S_4 is then presented.

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1 Introduction

For an arbitrary nonempty set S, define $A(S)$ to be the set of all one-to-one mapping of the set S onto itself. The set $A(S)$ with composition function operation is a group. If the set S contains n elements, then group $A(S)$ are denoted by S_n. Group S_n has $n!$ elements and will be called the symmetric group. There are many references on subgroups of S_2 and S_3 ([2],[4] and [5]). In this paper, we determine all subgroups of S_4 and then draw diagram of lattice subgroups of S_4.

One of the most important problem of fuzzy group theory is to classify the fuzzy subgroup of a finite group. Many papers have treated the particular case of finite cyclic group. Laszlo [3] studied about the construction of fuzzy subgroup of group of order one to six. Zhang and Zou [13] have determined the number of fuzzy subgroup of cyclic groups of order p^n where p is a prime number. Murali and Makamba ([11],[12]) considered a similar problem. They have determined the number of fuzzy subgroups of Z_{p^n},$Z_p \times Z_p$,$Z_{p^n} \times Z_q$ and abelian groups of order $p^n q^m$ where p and q are different primes. Tarnauceanu and Bentea [6] have determined this number for finite abelian groups. Sulaiman and Abd Ghafur [8] have counted the number of fuzzy subgroups of finite cyclic groups. All of them have treated the particular case of finite abelian group. It is interesting to investigate the number of fuzzy subgroups of nonabelian groups. Sulaiman and Abd Ghafur [7] have counted the number of fuzzy subgroups of nonabelian symmetric groups S_2, S_3 and alternating group A_4. They [9] have counted for group defined by a presentation, too.

There is a very close relationship between the number of chain of lattice subgroup of group G and the number of fuzzy subgroup of G. Therefore, the result of this paper, that is a diagram of lattice subgroups of S_4 is very important to determine the number of fuzzy subgroup of S_4.

2 Preliminary

We recall some definitions and results that will be used later.

Definition 2.1 *A partial ordered on a nonempty set P is a binary relation $\leq$ on P that is reflexive, antisymmetric and transitive. The pair $< P, \leq >$ is called a partially ordered set or poset. Poset $< P, \leq >$ is totally ordered if every $x, y \in P$ are comparable, that is $x \leq y$ or $y \leq x$. A nonempty subset S of P is a chain in P if S is totally ordered by $\leq$. (Roman, [10])*

Definition 2.2 *Let $< P, \leq >$ be a poset and let $S \subseteq P$. An upper bound for S is an element $x \in P$ for which $s \leq x, \forall s \in S$. The least upper bound of S is called the supremum or join of S. A lower bound for S is an element $x \in P$ for which $x \leq s, \forall s \in S$. The greatest lower bound of S is called the infimum or meet of S. Poset $< P, \leq >$ is called a lattice if every pair x, y elements of P has a supremum and an infimum (Roman, [10]).*

Note that the set of all of subgroups G under the "subgroup" relation is a lattice. This lattice is called the lattice subgroup of G.

Theorem 2.3 *(Lagrange Theorem) If G is a finite group and H is a subgroup of G, then order of H is a divisor of order G (Herstein, [4]).*

Theorem 2.4 *If G is a finite group and $a \in G$, then order of a is a divisor of order G (Herstein, [4]).*

Theorem 2.5 *(The First Sylow Theorem) Let G be a finite group and let $|G| = p^n m$ where $n \geq 1$, p is a prime number and $(p, m) = 1$. Then G contains a subgroup of order p^i for each i where $1 \leq i \leq n$ (Fraleigh, [5]).*

Definition 2.6 *Let G be a finite group and let $|G| = p^n m$ where $n \geq 1$, p is a prime number and $(p, m) = 1$. The subgroup of G of order p^n is called the sylow $p-$subgroup of G (Gardiner, [2]).*

Theorem 2.7 *(The Third Sylow Theorem) Let G be a finite group and let $|G| = p^n m$ where $n \geq 1$, p is a prime number and $(p, m) = 1$. Then the number of Sylow $p-$subgroup is of the form $(1+kp)$, where k is a non-negative integer, and $(1 + kp)$ divides the order of G (Fraleigh, [5]).*

Definition 2.8 *A subgroup N of G is said to be a normal subgroup of G if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$ (Herstein, [4]).*

Theorem 2.9 *There is a unique Sylow $p-$subgroup of the finite group G if only if it is normal (Gardiner, [2]).*

Theorem 2.10 *Let G be a group of order pq, where p and q are distinct primes and $p < q$. Then G has only one subgroup of order q. This subgroup of order q is normal in G (Gardiner, [2]).*

3 Main Results

Let $S_4 = \{i = id, \sigma_1, \sigma_2, ..., \sigma_9, \tau_1, \tau_2, ..., \tau_8, \alpha_1, \alpha_2, ..., \alpha_6\}$ with $\sigma_1 = (1234)$,$\sigma_2 = (13)(24)$, $\sigma_3 = (1432)$, $\sigma_4 = (1243)$,$\sigma_5 = (14)(23)$, $\sigma_6 = (1342)$, $\sigma_7 = (1324)$, $\sigma_8 = (12)(34)$, $\sigma_9 = (1432)$,$\tau_1 = (234)$,$\tau_2 = (243)$, $\tau_3 = (134)$, $\tau_4 = (143)$, $\tau_5 = (124)$, $\tau_6 = (142)$, $\tau_7 = (123)$, $\tau_8 = (132)$, $\alpha_1 = (12)$,$\alpha_1 = (12)$,$\alpha_2 = (13)$, $\alpha_3 = (14)$, $\alpha_4 = (23)$, $\alpha_5 = (24)$, $\alpha_6 = (34)$. Multiplication table of S_4 can be seen in [1]. The order of elements of S_4 are shown as follows.

Order	Elements
1	i
2	σ_2, σ_5, σ_8, α_1, α_2, α_3, α_4, α_5, α_6
3	τ_1, τ_2, τ_3, τ_4, τ_5, τ_6, τ_7, τ_8
4	σ_1, σ_3, σ_4, σ_6, σ_7, σ_9

According to the Lagrange theorem, nontrivial subgroups of S_4 must have order 2, 4, 6, 8 or 12. We will determine all of the subgroups of S_4. Clearly, the subgroup of S_4 of order 1 is the trivial subgroup $H_1 = \{i\}$.

Subgroups of order 2:
Let K be arbitrary subgroup of S_4 of order 2. Since 2 is a prime number, then K is cyclic. Therefore K is generated by an element of S_4 of order 2. Thus, all of subgroups of S_4 of order 2 are $K_2 = \{i, \sigma_2\}$, $K_3 = \{i, \sigma_5\}$, $K_4 = \{i, \sigma_8\}$, $K_5 = \{i, \alpha_1\}$, $K_6 = \{i, \alpha_2\}$, $K_7 = \{i, \alpha_3\}$, $K_8 = \{i, \alpha_4\}$, $K_9 = \{i, \alpha_5\}$, $K_{10} =$

$\{i, \alpha_6\}$.

Subgroups of order 3:
Similar to order 2, subgroups of S_4 of order 3 is generated by an element of S_4 of order 3. Thus, all of subgroups of S_4 of order 3 are $L_{11} =< \tau_1 >=< \tau_2 >= \{i, \tau_1, \tau_2\}$, $L_{12} =< \tau_3 >=< \tau_4 >= \{i, \tau_3, \tau_4\}$, $L_{13} =< \tau_5 >=< \tau_6 >= \{i, \tau_5, \tau_6\}$, $L_{14} =< \tau_7 >=< \tau_8 >= \{i, \tau_7, \tau_8\}$.

Subgroups of order 4:
Let M be any arbitrary subgroup of S_4 of order 4. According to the Theorem 2.4, then elements of M must have order 1, 2 or 4. If M consists an element of order 4, then M is generated by an element of order 4. The subgroups are $M_{15} =< \sigma_1 >=< \sigma_2 >=< \sigma_3 >= \{i, \sigma_1, \sigma_2, \sigma_3\}$, $M_{16} =< \sigma_4 >=< \sigma_5 >=< \sigma_6 >= \{i, \sigma_4, \sigma_5, \sigma_6\}$, $M_{17} =< \sigma_7 >=< \sigma_8 >=< \sigma_9 >= \{i, \sigma_7, \sigma_8, \sigma_9\}$.
If M consists an element of order 2, then M does not have element of order 4. Therefore the order of all of the elements of M is 2, except identity element. We will try to multiply all of the combination of elements of order 2. The following is the multiplication table of the elements of order 2.

	σ_2	σ_5	σ_8	α_1	α_2	α_3	α_4	α_5	α_6
σ_2	i	σ_8	σ_5	σ_7	α_5	σ_6	σ_4	α_2	σ_9
σ_5	σ_8	i	σ_2	σ_9	σ_3	α_4	α_3	σ_1	σ_7
σ_8	σ_5	σ_2	i	α_6	σ_1	σ_4	σ_6	σ_3	α_1
α_1	σ_9	σ_7	α_6	i	τ_7	τ_5	τ_8	τ_6	σ_8
α_2	α_5	σ_1	σ_3	τ_8	i	τ_3	τ_7	σ_2	τ_4
α_3	σ_4	α_4	σ_6	τ_6	τ_4	i	σ_5	τ_5	τ_3
α_4	σ_2	α_3	σ_4	τ_7	τ_8	σ_5	i	τ_1	τ_2
α_5	α_2	σ_3	σ_1	τ_5	σ_2	τ_6	τ_2	i	τ_1
α_6	σ_7	σ_9	α_1	σ_8	τ_3	τ_4	τ_1	τ_2	i

We observe the table above by row, from the second row. We conclude that $\{\sigma_2, \alpha_1\}, \{\{\sigma_2, \alpha_3\}, \{\sigma_2, \alpha_4\}, \{\sigma_2, \alpha_6\}$ are not subsets of M, because the order of the result of their product is not 2. We get the subgroups of order 4 containing σ_2 are $M_{18} = \{i, \sigma_2, \sigma_5, \sigma_8\}$ and $M_{19} = \{i, \sigma_2, \alpha_2, \alpha_5\}$. Using similar argument to the other rows, we obtain the other subgroups of order 4, those are $M_{20} = \{i, \sigma_5, \alpha_3, \alpha_4\}$ and $M_{21} = \{i, \sigma_8, \alpha_1, \alpha_6\}$. Thus, we got seven subgroups of order 4.

Subgroups of order 6:
Let N be any arbitrary subgroup of S_4 of order 6. Since $|N| = 2 \times 3$, according to the Theorem 2.7 (The Third Sylow Theorem) then N has only one subgroup M of order 3. According to the Theorem 2.9, this subgroup M is normal in N . There are four possibilities subgroups of order 3, those are L_{11}, L_{12}, L_{13} or L_{14}.

Using normality M in N, we got four subgroups of order 6, those are $N_{22} = \{i, \tau_1, \tau_2, \alpha_4, \alpha_5, \alpha_6\}, N_{23} = \{i, \tau_3, \tau_4, \alpha_2, \alpha_3, \alpha_6\}, N_{24} = \{i, \tau_5, \tau_6, \alpha_1, \alpha_3, \alpha_5\}$ and $N_{25} = \{i, \tau_7, \tau_8, \alpha_1, \alpha_2, \alpha_4\}$.

Subgroups of order 8:
Let P be arbitrary subgroup of S_4 of order 8. Since there is no element of S_4 of order 8, then every element of P must have order 1, 2 or 4. Also, since $|S_4| = 2^3 3$, then by Theorem 2.7, the number of subgroups of S_4 of order 8 is 1 or 3. We will try to construct all of the possibilities and we will stop if we get three subgroups. We start from the table of orbit of element of order 4.

Element	Orbit
σ_1, σ_3	$\{\sigma_1, \sigma_2, \sigma_3, i\}$
σ_4, σ_6	$\{\sigma_4, \sigma_5, \sigma_6, i\}$
σ_7, σ_9	$\{\sigma_7, \sigma_8, \sigma_9, i\}$

Considering the orbit, we can conclude that P have only three possibilities:
i) Consists two elements of order 4
ii) Consists four elements of order 4
iii) No element of order 4.
First, we will determine subgroup of order 8 containing two elements of order 4. The following is a multiplication table between elements of order 4 (which have same orbit) and elements of order 2.

	σ_2	σ_5	σ_8	α_1	α_2	α_3	α_4	α_5	α_6
σ_1	σ_3	α_2	α_5	τ_1	σ_8	τ_7	τ_3	σ_5	τ_5
σ_3	σ_1	α_5	α_2	τ_4	σ_5	τ_2	τ_6	σ_8	τ_8

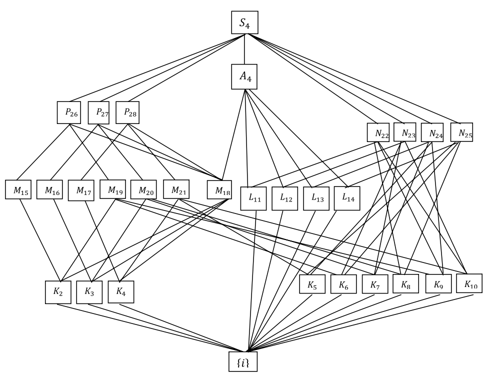
Considering the table above, we conclude that $\alpha_1, \alpha_3, \alpha_4$ or α_6 can not be added to $\{i, \sigma_1, \sigma_1\sigma_3\}$ in order to be a subgroup of order 8. We get a subgroup of order 6 if we add the other elements of order 2 namely, $P_{26} = \{i, \sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_8, \alpha_2, \alpha_5\}$. Using similar argument, we get the other subgroups of S_4 of order 8, those are $P_{27} = \{i, \sigma_2, \sigma_4, \sigma_5, \sigma_6, \sigma_8, \alpha_3, \alpha_4\}$ and $P_{28} = \{i, \sigma_2, \sigma_5, \sigma_7, \sigma_8, \sigma_9, \alpha_1, \alpha_6\}$.

Subgroups of order 12:
Obviously the alternating group $A_4 = \{i, \sigma_2, \sigma_5, \sigma_8, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8\}$ is a subgroup of S_4 of order 12. We will prove that A_4 is the unique subgroup of S_4 of order 12. Let $H \leq S_4$ where $|H| = 12$ and $H \neq A_4$. We conclude that (from Theorem 2.7), there are three possibilities:
i) The number of subgroups H of order 4 is 3 and of order 3 is 4
ii) The number of subgroups H of order 4 is 1 and of order 3 is 1

iii) The number of subgroups H of order 4 is 3 and of order 3 is 1.

The case i) cannot occur, because three subgroups of order 3 together with four subgroups of order 4 consist more than 12 elements.
Case ii): Let the subgroup of order 3 is K and let the subgroup of order 4 is L. According to the Theorem 5, then K and L are normal in H. We can show that $H = KL$ and $kl = lk, \forall k \in K, l \in L$. Hence, H must be abelian. Checking all of possibilities of K and L, we get a conclusion that this case cannot occur.
Case iii): Let $\mathcal{A}$ is a collection of the three subgroups of order 4 and K is the subgroup of order 3. According to the Theorem 5, then K is normal in H. Therefore $h^{-1}kh \in K, \forall h \in H, k \in K$. Giving counter examples for every possibility of K we conclude this condition is not held for every subgroup K of order 3. This means that this case cannot occur. Therefore we got only one subgroup of order 12 namely alternating group, A_4.

According to this result, we have the diagram of lattice subgroups of S_4 is shown as figure.1 below.



1

Figure 1: Subgroup lattice of S_4.

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