A Note on Nilpotent Elements in Quaternion Rings over $\mathbb{Z}_p$

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Abstract
In this paper, we discuss nilpotent elements in the finite ring $\mathbb{H}/\mathbb{Z}_p$. We provide examples and we establish conditions for nilpotency in $\mathbb{H}/\mathbb{Z}_p$. We also comment on the number of nilpotent elements.

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1. Introduction
The quaternions, denoted by $\mathbb{H}$, were first invented by W. R. Hamilton in 1843 as an extension of the complex numbers into four dimensions [8]. Algebraically speaking, $\mathbb{H}$ forms a division algebra (skew field) over $\mathbb{R}$ of dimension 4 ([8], p.195-196). In [1], we studied the finite ring $^1\mathbb{H}/\mathbb{Z}_p$, where $p$ is a prime, looking into its structure and some of its properties. A more detailed description of the structure $\mathbb{H}/\mathbb{Z}_p$ was given by Miguel and Serodio in [5]. Among others, they found the number of zero-divisors, the number of idempotent elements, and provided an interesting description of the zero-divisor graph. In particularly, they showed that the number of idempotent elements in $\mathbb{H}/\mathbb{Z}_p$ is $p^2 + p + 2$,
for $p$ odd prime. In [2], we studied idempotent elements and gave conditions for idempotency. In the sections that follow, we examine nilpotent elements in $\mathbb{H}/\mathbb{Z}_p$ and provide conditions for nilpotency in $\mathbb{H}/\mathbb{Z}_p$.

2. Nilpotent Elements in $\mathbb{H}/\mathbb{Z}_p$

Recall that an element $x$ in a ring $R$ is called nilpotent if $x^k = 0$, for some $k \in \mathbb{Z}$. In the ring $\mathbb{H}/\mathbb{Z}_p$, $p$ prime, an element $x$ is of the form:

$$x = a_0 + a_1 i + a_2 j + a_3 k$$

where $a_i \in \mathbb{Z}_p$, $p$ prime, and $i^2 = j^2 = k^2 = p - 1 = -1$. It is not hard to show that for any $x \in \mathbb{H}/\mathbb{Z}_p$, one has:

$$x^2 - 2a_0 x + N(x) = 0$$

(1)

where $N(x)$ is the norm of $x$. Namely, $N(x) = a_0^2 + a_1^2 + a_2^2 + a_3^2$. Considering (1), we have now the following Lemma:

**Lemma 2.1:** Let $x \in \mathbb{H}/\mathbb{Z}_p$. If $x$ is nilpotent, then $N(x) = 0$.

**Proof.** If $x$ is nilpotent, then $x^k = 0$ for some $k$. From (1) above, we have:

$$x^2 - 2a_0 x + N(x) = 0 \Rightarrow x(x - 2a) = -N(x)$$

$$\Rightarrow x^k(x - 2a)^k = -N(x)^k$$

$$\Rightarrow 0 = N(x)^k$$

$$\Rightarrow N(x) = 0$$

because $\mathbb{Z}_p$ is a field. \qed

**Theorem 2.2:** Let $x \in \mathbb{H}/\mathbb{Z}_p$. If $x$ is nilpotent, then $x = a_1 i + a_2 j + a_3 k$ with $N(x) = 0$.

Furthermore, $x^2 = 0$.

**Proof.** From (1) and Lemma 2.1, we have:

(i) $k = $ even:

$$x^2 = 2ax \Rightarrow (x^2)^{k/2} = (2a_0)^{k/2}x^{k/2}$$

$$\Rightarrow x^k = (2a_0)^{k/2}x^{k/2}$$

$$\Rightarrow 0 = (2a_0)^{k/2}x^{k/2}$$

$$\Rightarrow a_0 = 0.$$
(ii) $k = \text{odd}$:

$$x^2 = 2ax \Rightarrow (x^2)^{(k+1)/2} = (2a_0)^{(k+1)/2}x^{(k+1)/2}$$
$$\Rightarrow x^{k+1} = (2a_0)^{(k+1)/2}x^{(k+1)/2}$$
$$\Rightarrow 0 = (2a_0)^{(k+1)/2}x^{(k+1)/2}$$
$$\Rightarrow a_0 = 0.$$

The fact that $x^2 = 0$ follows immediately. $\square$

**Example 2.3:** Let $p = 3$. Then, $x = 2i + 4j + 2k$ is nilpotent. Notice that $N(x) = 0$ and $x^2 = 0$.

**Remark 2.4:** Regarding the number of nilpotent elements, Fine and Herstein in [3] show that the probability that an $n \times n$ matrix over a Galois field having $p^\alpha$ elements be nilpotent is $p^{-\alpha.n}$. In our case, $n = 2$ and $\alpha = 1$. So, the probability that a $2 \times 2$ matrix over $\mathbb{Z}_p$ be nilpotent is $p^{-2}$. In other words, the $(\# \text{ of } 2 \times 2 \text{ nilpotent matrices})/\text{(\# of } 2 \times 2 \text{ matrices}) = p^{-2}$. Hence, $(\# \text{ of } 2 \times 2 \text{ nilpotent matrices})/p^4 = p^{-2}$. Which gives that the $(\# \text{ of } 2 \times 2 \text{ nilpotent matrices}) = p^2$.

**Conclusion**

We discussed nilpotent elements in $\mathbb{H}/\mathbb{Z}_p$ and gave conditions for their existence, as well as some examples. An interesting and possibly harder project is to look at the structure of $\mathcal{O}/\mathbb{Z}_p$, where $\mathcal{O}$ is the octonion division algebra, and discuss idempotent and nilpotent elements in that finite ring.

**Notes**

1. Recall that addition and multiplication on $\mathbb{H}/\mathbb{Z}_n$ are defined as follows:

$$x + y = (a_0 + a_1i + a_2j + a_3k) + (b_0 + b_1i + b_2j + b_3k)$$
$$= (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k$$

$$x \cdot y = (a_0 + a_1i + a_2j + a_3k) \cdot (b_0 + b_1i + b_2j + b_3k)$$
$$= a_0b_0 + (n-1)a_1b_1 + (n-1)a_2b_2 + (n-1)a_3b_3 +$$
$$+ (a_0b_1 + a_1b_0 + a_2b_3 + (n-1)a_3b_2)i +$$
$$+ (a_0b_2 + (n-1)a_1b_3 + a_2b_0 + a_3b_1)j +$$
$$+ (a_0b_3 + a_1b_2 + (n-1)a_2b_1 + a_3b_0)k$$
2. Over $\mathbb{H}$ the same equation holds. See Exercise 37(g) (p.214) in [6].

3. If $x$ is idempotent, then $x = \frac{1}{2} a_0 + a_1 i + a_2 j + a_3 k$, with $N(x) = 0$. This is essentially our Theorem 2.5 in [2] (putting $p = 0$). Over $\mathbb{H}$ the same equation holds. See Exercise 37(g) (p.214) in [6].

References


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