Ordering Generalized Trapezoidal Fuzzy Numbers

Using Orthocentre of Centroids

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Abstract

Ordering fuzzy numbers plays an important role in approximate reasoning, optimization, forecasting, decision making, controlling, scheduling and other usage. This paper illustrates a ranking method for ordering fuzzy numbers based on Area, Mode, Spreads and Weights of generalized (non-normal) fuzzy numbers. The area used in this method is obtained from the non-normal trapezoidal fuzzy number, first by splitting the generalized trapezoidal fuzzy numbers into three plane figures and then calculating the Centroids of each plane figure followed by the Orthocentre of these Centroids and then finding the area of this Orthocentre from the original point. In this paper, we also apply mode and spreads in those cases where the discrimination is not possible. This method is simple in evaluation and can rank various types of fuzzy numbers and also crisp numbers which are considered to be a special case of fuzzy numbers.

Keywords: Ranking function; Orthocentre; Centroid points; Generalized trapezoidal fuzzy numbers

1. Introduction

Ranking fuzzy numbers plays an important role in decision making. Most of the real world problems that exist in nature are fuzzy, than probabilistic or deterministic. Problems in which fuzzy theory is used, like fuzzy risk analysis, fuzzy optimization, etc., at one or other stage fuzzy numbers must be ranked before an action is taken by a decision maker. As fuzzy numbers are represented by possibility distributions they often overlap with each other and discriminating them is a complex task than discriminating real numbers where a natural order exist between them. An efficient approach for ordering the fuzzy numbers is defuzzification. For this we define a ranking function from the set of all fuzzy
numbers $F(R)$ to the set of all real numbers ‘$R$’, which maps each fuzzy number into the real line, where a natural order exists. Usually by reducing the whole of any analysis to a single number, much of the information is lost and most of the ranking methods consider only one point of view on comparing fuzzy quantities. Hence an attempt is to be made to minimize this loss.

Since the inception of fuzzy sets by Zadeh [23] in 1965, many authors have proposed different methods for ranking fuzzy numbers. However, due to the complexity of the problem, there is no method which gives a satisfactory result to all situations. Most of the methods proposed so far are non-discriminating, counter-intuitive and some produce different rankings for the same situation and some methods cannot rank crisp numbers. Ranking fuzzy numbers was first proposed by Jain in the year 1976 for decision making in fuzzy situations by representing the ill-defined quantity as a fuzzy set. Jain [13, 14] proposed a method using the concept of maximizing set to order the fuzzy numbers and the decision maker considers only the right side membership function. Since then, various procedures to rank fuzzy quantities are proposed by various researchers. Yager [21, 22] proposed four indices to order fuzzy quantities in $[0, 1]$. An Adamo [2] fuzzy decision tree was an important breakthrough in ranking fuzzy numbers. Dubois and Prade [10] proposed a complete set of comparison indices in the frame work of Zadeh’s possibility theory. Bortolan and Degani [3] reviewed some of these ranking methods for ranking fuzzy subsets. Chen [5] presented ranking fuzzy numbers with maximizing set and minimizing set. Kim and Park [15] presented a method of ranking fuzzy numbers with index of optimism. Liou and Wang [17] presented ranking fuzzy numbers with integral value. Choobineh and Li [8] presented an index for ordering fuzzy numbers. Since then several methods have been proposed by various researchers which include ranking fuzzy numbers using area compensation by Fortemps and Roubens [11], distance method by Cheng [7]. Wang and Kerre [19, 20] classified the existing ranking procedures into three classes. The first class consists of ranking procedures based on fuzzy mean and spread and second class consists ranking procedures based on fuzzy scoring whereas, the third class consists of methods based on preference relations and concluded that the ordering procedures associated with first class are relatively reasonable for the ordering of fuzzy numbers specially, the ranking procedure presented by Adamo [2] which satisfies all the reasonable properties for the ordering of fuzzy quantities. The methods presented in the second class are not doing well and the methods which belong to class three are reasonable. Later on, ranking fuzzy numbers by area between the centroid point and original point by Chu and Tsao [9], modification of the index of Liou and Wang by Garcia and Lamata [12], fuzzy risk analysis based on ranking of generalized trapezoidal fuzzy numbers by Chen and Chen [4], a new approach for ranking trapezoidal fuzzy numbers by Abbasbandy and Hajjari [1], fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads by Chen and Chen [6] came into existence. Amit Kumar et al. [16] presented a procedure on ranking generalized trapezoidal fuzzy numbers based on rank, mode, divergence and spread. Rao and Shankar
[18] presented a method on ranking fuzzy numbers using circumcenter of centroids and index of modality. In this paper a new method is proposed which is based on Orthocentre of Centroids to rank fuzzy quantities. In a trapezoidal fuzzy number, first the trapezoid is split into three parts where the first, second and third parts are a triangle, a rectangle and a triangle respectively. Then the centroids of these three parts are calculated followed by the calculation of the Orthocentre of these centroids. Finally, a ranking procedure is defined which is the area between the Orthocentre of Centroids and the original point and also uses mode and spreads in those cases where the discrimination is not possible.

In section 2 , we briefly introduce the basic definitions, arithmetic operations of fuzzy numbers and some fuzzy ranking methods. Section 3 presents the proposed ranking method. In Section 4, some important results like linearity of ranking function and other properties are proved which are useful for proposed approach. In Section 5, the proposed method has been explained with examples which describe the advantages and the efficiency of the method. In Section 6, the method demonstrates its power by comparing with other methods that exist in literature. Finally, the conclusions of the work are presented in Section 7.

2. Basic definitions, fuzzy arithmetic and ranking of fuzzy numbers

2.1 Basic Definitions

Definition 1. Let \( U \) be a universe set. A fuzzy set \( \tilde{A} \) of \( U \) is defined by a membership function \( f_A : U \rightarrow [0,1] \), where \( f_A(x) \) is the degree of \( x \) in \( \tilde{A} \), \( \forall x \in U \).

Definition 2. A fuzzy set \( \tilde{A} \) of universe set \( U \) is normal if and only if \( \sup_{x \in U} f_A(x) = 1 \).

Definition 3. A fuzzy set \( \tilde{A} \) of universe set \( U \) is convex if and only if
\[
f_A(\lambda x + (1 - \lambda) y) \geq \min\left(f_A(x), f_A(y)\right), \forall x, y \in U \text{ and } \lambda \in [0,1].
\]

Definition 4. A fuzzy set \( \tilde{A} \) of universe set \( U \) is a fuzzy number iff \( \tilde{A} \) is normal and convex on \( U \).

Definition 5. A real fuzzy number \( \tilde{A} \) is described as any fuzzy subset of the real line \( A \) with membership function \( f_A(x) \) possessing the following properties:

(1) \( f_A(x) \) is a continuous mapping from \( \mathbb{R} \) to the closed interval \([0,w]0 < w \leq 1\).
(2) \( f_A^x(x) = 0 \), for all \( x \in (-\infty, a] \)

(3) \( f_A^x(x) \) is strictly increasing on \([a, b]\)

(4) \( f_A^x(x) = 1 \), for all \( x \in [b, c] \)

(5) \( f_A^x(x) \) is strictly decreasing on \([c, d]\)

(6) \( f_A^x(x) = 0 \), for all \( x \in (d, \infty] \), where \( a, b, c, d \) are real numbers

**Definition 6.**

The membership function of the real fuzzy number \( \tilde{A} \) is given by

\[
f_A^x(x) = \begin{cases} 
  f_A^L, & a \leq x \leq b, \\
  w, & b \leq x \leq c, \\
  f_A^K, & c \leq x \leq d, \\
  0, & \text{otherwise,}
\end{cases}
\]

where \( 0 < w \leq 1 \) is a constant, \( a, b, c, d \) are real numbers and \( f_A^L : [a, b] \to [0, w] \), \( f_A^K : [c, d] \to [0, w] \) are two strictly monotonic and continuous functions from \( \mathbb{R} \) to the closed interval \([0, w]\). It is customary to write a fuzzy number as \( \tilde{A} = (a, b, c, d; w) \). If \( w = 1 \), then \( \tilde{A} = (a, b, c, d; 1) \) is a normalized fuzzy number, otherwise \( \tilde{A} \) is said to be a generalized or non-normal fuzzy number.

If the membership function \( f_A^x(x) \) is piecewise linear, then \( \tilde{A} \) is said to be a trapezoidal fuzzy number. The membership function of a trapezoidal fuzzy number is given by:

\[
f_A^x(x) = \begin{cases} 
  \frac{w(x-a)}{b-a}, & a \leq x \leq b, \\
  w, & b \leq x \leq c, \\
  \frac{w(x-d)}{c-d}, & c \leq x \leq d, \\
  0, & \text{otherwise,}
\end{cases}
\]

If \( w = 1 \), then \( \tilde{A} = (a, b, c, d; 1) \) is a normalized trapezoidal fuzzy number and \( \tilde{A} \) is a generalized or non-normal trapezoidal fuzzy number if \( 0 < w < 1 \).

The image of \( \tilde{A} = (a, b, c, d; w) \) is given by \( -\tilde{A} = (-d, -c, -b, -a; w) \).
As a particular case if \( b = c \), the trapezoidal fuzzy number reduces to a triangular fuzzy number given by \( \tilde{A} = (a, b, d; w) \). The value of ‘\( b \)’ corresponds with the mode or core and \([a, d]\) with the support. If \( w=1 \), then \( A = (a, b, d) \) is a normalized triangular fuzzy number. \( \tilde{A} \) is a generalized or non normal triangular fuzzy number if \( 0 < w < 1 \).

### 2.2 Fuzzy arithmetic operations

If \( \tilde{A} = (a_1, b_1, c_1, d_1; w_1) \) and \( B = (a_2, b_2, c_2, d_2; w_2) \) are two generalized trapezoidal fuzzy numbers, then

(i) \( \tilde{A} \oplus \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \min(w_1, w_2)) \)

(ii) \( \tilde{A} \ominus \tilde{B} = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; \min(w_1, w_2)) \)

(iii) \( k \tilde{A} = (ka_1, kb_1, kc_1, kd_1; w_1); k > 0 \)

(iv) \( k \tilde{A} = (kd_1, kc_1, kb_1, ka_1; w_1); k < 0 \).

### 2.3 Some methods of ranking fuzzy numbers

In this section, some methods of ranking fuzzy numbers are reviewed.

#### Yager fuzzy ranking

Yager [22] proposed a centroid-index ranking method for calculating the value \( \mathbf{x}^\ast \) for a fuzzy number \( \tilde{A} \) as follows:

\[
\mathbf{x}^\ast = \frac{\int \mathbf{w}(x) f_\tilde{A}(x) dx}{\int f_\tilde{A}(x) dx},
\]

where \( w(x) \) denotes the value of a weighing function measuring the importance of the value \( x \) and \( f_\tilde{A}(x) \) denotes the membership function of the fuzzy number \( \tilde{A} \), where \( f_\tilde{A}: \mathbb{X} \to [0,1] \). When \( w(x) = x \), the value \( \mathbf{x}^\ast \) shown in Eq (3) becomes the geometric center of gravity (COG) [22] shown as follows:

\[
\mathbf{x}^\ast = \frac{\int x f_\tilde{A}(x) dx}{\int f_\tilde{A}(x) dx}
\]

(4)

The larger the value of \( \mathbf{x}^\ast \), the better the ranking of \( \tilde{A} \).
Cheng’s fuzzy ranking
Cheng [7] ranked fuzzy numbers with the distance method using the Euclidean distance between the Centroid point and original point. For a generalized fuzzy number \( \tilde{A} = (a, b, c, d; w) \), the centroid is given by:

\[
(x_0, y_0) = \left( \frac{w(d^2 - 2c^2 + 2b^2 - a^2 + dc - ab) + 3(c^2 - b^2)}{3w(d - c + b - a) + 6(c - b)}, \frac{w}{3}\left(1 + \frac{(b + c) - (a + d)(1 - w)}{(b + c - a - d) + 2(a + d)w}\right) \right)
\]

and the ranking function \( \tilde{A}_i > \tilde{A}_j \) associated with \( \tilde{A} \) as \( R(\tilde{A}) = \sqrt{x_0^2 + y_0^2} \)

Let \( \tilde{A}_i \) and \( \tilde{A}_j \) two fuzzy numbers, (i) If \( R(\tilde{A}_i) > R(\tilde{A}_j) \) then \( \tilde{A}_i > \tilde{A}_j \).

(ii) If \( R(\tilde{A}_i) < R(\tilde{A}_j) \) then \( \tilde{A}_i < \tilde{A}_j \) (iii) If \( R(\tilde{A}_i) = R(\tilde{A}_j) \) then \( \tilde{A}_i = \tilde{A}_j \).

He further improved Lee and Li’s method by proposing the index of coefficient of variation (CV) as \( CV = \frac{\sigma}{\mu} \) where \( \sigma \) is standard error and \( \mu \) is mean, \( \mu \neq 0 \) and \( \sigma > 0 \), the fuzzy number with smaller CV is ranked higher.

Liou and Wang’s fuzzy ranking
Liou and Wang [17] ranked fuzzy numbers with total integral value. For a fuzzy number \( \tilde{A} \), the total integral value is defined as \( I_\alpha(\tilde{A}) = \alpha I_R(\tilde{A}) + (1 - \alpha)I_L(\tilde{A}) \) where

\[
I_R(\tilde{A}) = \int_0^1 g_\tilde{A}^R(y)dy \quad \text{and} \quad I_L(\tilde{A}) = \int_0^1 g_\tilde{A}^L(y)dy
\]

are the right and left integral values of \( \tilde{A} \), respectively and \( \alpha \in [0,1] \) is the index of optimism which represents the degree of optimism of a decision maker. If \( \alpha = 0 \) the total integral value represents a pessimistic decision maker’s view point which is equal to left integral value. If \( \alpha = 1 \), the total integral value represents an optimistic decision maker’s view point and is equal to the right integral value and when \( \alpha = 0.5 \), the total integral value represents an moderate decision maker’s view point and is equal to the mean of right and left integral values. For a decision maker, the larger the value of \( \alpha \) is, the higher is the degree of optimism.

3. Proposed ordering of fuzzy numbers
The Centroid of a trapezoid is measured as the balancing point of the trapezoid (Fig.1). Divide the trapezoid into three plane figures. These three plane figures are a triangle (APB), a rectangle (BPQC), and a triangle (CQD), respectively. Let the Centroids of the three plane figures be \( G_1, G_2, \) & \( G_3 \) respectively. The
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The orthocenter of these Centroids $G_1, G_2, G_3$ is taken as the point of position to define the ranking of generalized trapezoidal fuzzy numbers. The reason for selecting this point as a point of position is that each Centroid point are balancing points of each individual plane figure, and the orthocentre of these Centroid points is a much more balancing point for a generalized trapezoidal fuzzy number. Therefore, this point would be a better position point than the Centroid point of the trapezoid.

Consider a generalized trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$. (Fig.1). The Centroids of the three plane figures are $G_1 = \left(\frac{a + 2b}{3}, \frac{w}{3}\right)$, $G_2 = \left(\frac{b + c}{2}, \frac{w}{2}\right)$ and $G_3 = \left(\frac{2c + d}{3}, \frac{w}{3}\right)$ respectively. Equation of the line $G_1G_3$ is $y = \frac{w}{3}$ and $G_2$ does not lie on the line $G_1G_3$. Therefore, $G_1, G_2$ and $G_3$ are non-collinear and they form a triangle.

We define the Orthocentre $O_\tilde{A}(x_0, y_0)$ of the triangle with vertices $G_1, G_2$ and $G_3$ of the generalized trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$ as

$$O_\tilde{A}(x_0, y_0) = \left(\frac{1}{2}(c + b), \frac{(3c - 2a - b)(c - 3b + 2d) + 2w^2}{6w}\right)$$  \hspace{1cm} (5)

As a special case, for triangular fuzzy number $\tilde{A} = (a, b, d; w)$, i.e., $c = b$ the Orthocentre of Centroids is given by

$$O_\tilde{A}(x_0, y_0) = \left(\frac{b}{3}, \frac{(b - a)(d - b) + w^2}{3w}\right)$$  \hspace{1cm} (6)

The ranking function of the generalized trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of real numbers is defined as:
This is the Area between the Orthocentre of the Centroids \(O_A(x_o, y_o)\) as defined in Eq.(5) and the original point.

The Mode (m) of the generalized trapezoidal fuzzy number \(\tilde{A} = (a, b, c, d; w)\). is defined as:

\[
m = \frac{1}{2} \int_0^w (b + c) dx = \frac{w}{2} (b + c)
\]

The Spread(s) of the generalized trapezoidal fuzzy number \(\tilde{A} = (a, b, c, d; w)\). is defined as:

\[
s = \int_0^w (d - a) dx = w(d - a)
\]

The Left spread (ls) of the generalized trapezoidal fuzzy number \(\tilde{A} = (a, b, c, d; w)\). is defined as:

\[
ls = \int_0^w (b - a) dx = w(b - a)
\]

The Right spread (rs) of the generalized trapezoidal fuzzy number \(\tilde{A} = (a, b, c, d; w)\). is defined as:

\[
rs = \int_0^w (d - c) dx = w(d - c)
\]

Using the above definitions we now define the ranking procedure of two generalized trapezoidal fuzzy numbers.

Let \(\tilde{A} = (a_1, b_1, c_1, d_1; w_1)\) and \(\tilde{B} = (a_2, b_2, c_2, d_2; w_2)\) be two generalized trapezoidal fuzzy numbers. The working procedure to compare \(\tilde{A}\) and \(\tilde{B}\) is as follows:

Step 1: Find \(R(\tilde{A})\) and \(R(\tilde{B})\)

Case (i) If \(R(\tilde{A}) > R(\tilde{B})\) then \(\tilde{A} > \tilde{B}\)

Case (ii) If \(R(\tilde{A}) < R(\tilde{B})\) then \(\tilde{A} < \tilde{B}\)

Case (iii) If \(R(\tilde{A}) = R(\tilde{B})\) comparison is not possible, then go to step 2.

Step 2: Find \(m(\tilde{A})\) and \(m(\tilde{B})\)
Case (i) If \( \tilde{m}(\tilde{A}) > \tilde{m}(\tilde{B}) \) then \( \tilde{A} > \tilde{B} \)

Case (ii) If \( \tilde{m}(\tilde{A}) < \tilde{m}(\tilde{B}) \) then \( \tilde{A} < \tilde{B} \)

Case (iii) If \( \tilde{m}(\tilde{A}) = \tilde{m}(\tilde{B}) \) comparison is not possible, then go to step 3.

Step 3: Find \( s(\tilde{A}) \) and \( s(\tilde{B}) \)

Case (i) If \( s(\tilde{A}) > s(\tilde{B}) \) then \( \tilde{A} < \tilde{B} \)

Case (ii) If \( s(\tilde{A}) < s(\tilde{B}) \) then \( \tilde{A} > \tilde{B} \)

Case (iii) If \( s(\tilde{A}) = s(\tilde{B}) \) comparison is not possible, then go to step 4.

Step 4: Find \( ls(\tilde{A}) \) and \( ls(\tilde{B}) \)

Case (i) If \( ls(\tilde{A}) > ls(\tilde{B}) \) then \( \tilde{A} > \tilde{B} \)

Case (ii) If \( ls(\tilde{A}) < ls(\tilde{B}) \) then \( \tilde{A} < \tilde{B} \)

Case (iii) If \( ls(\tilde{A}) = ls(\tilde{B}) \) comparison is not possible, then go to step 5.

Step 5: Examine \( w_1 \) and \( w_2 \)

Case (i) If \( w_1 > w_2 \) then \( \tilde{A} > \tilde{B} \)

Case (ii) If \( w_1 < w_2 \) then \( \tilde{A} < \tilde{B} \)

Case (iii) If \( w_1 = w_2 \) then \( \tilde{A} \approx \tilde{B} \)

4. Some important results

In this section some important results which are the basis for defining the ranking procedure in section 3 are discussed and proved. The proposed ranking verifies all the reasonable properties[17] stated below:

Let \( S \) be an arbitrary finite subset of \( E \).

\( A_1 : \) For any arbitrary finite subset \( S \) of \( E \) and \( \tilde{A} \in S \),

\( \tilde{A} \geq \tilde{A} \)
A2 : $(\tilde{A}, \tilde{B}) \in S^2$, $\tilde{A} \geq \tilde{B}$ and $\tilde{B} \geq \tilde{A} \Rightarrow \tilde{A} \sim \tilde{B}$.

A3 : $(\tilde{A}, \tilde{B}, \tilde{C}) \in S^3$, $\tilde{A} \geq \tilde{B}$ and $\tilde{B} \geq \tilde{A} \Rightarrow \tilde{A} \geq \tilde{C}$.

A4 : $(\tilde{A}, \tilde{B}) \in S^2$, $\inf \{ \text{Supp} (\tilde{A}) \} \geq \sup \{ \text{Supp} (\tilde{B}) \} \Rightarrow \tilde{A} \geq \tilde{B}$.

A5 : Let $S$ and $S'$ be two arbitrary finite subsets of $E$ and $\tilde{A}, \tilde{B} \in S \cap S' \Rightarrow \tilde{A} \sim \tilde{B}$ on $S$ iff $\tilde{A} \sim \tilde{B}$ on $S'$.

A6 : Let $\tilde{A}, \tilde{B}, \tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C}$ be elements of $E$. If $\tilde{A} \geq \tilde{B}$, then $\tilde{A} \oplus \tilde{C} \geq \tilde{B} \oplus \tilde{C}$, where $\tilde{C} \neq \tilde{O}$.

Proposition 4.1 The ranking function defined in section 3 by means of Eq. (7) is a linear function for normalized trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; l)$. i.e. If $\tilde{A} = (a_1, b_1, c_1, d_1; l)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; l)$ are two normalized trapezoidal fuzzy numbers, then

(i) $R\left( k_1 \tilde{A} \oplus k_2 \tilde{B} \right) = k_1 R(\tilde{A}) \oplus k_2 R(\tilde{B})$; $k_1, k_2$ are non negative real numbers

(ii) $R(-\tilde{A}) = -R(\tilde{A})$

(iii) $R(\tilde{A}) \oplus (-\tilde{A}) = 0$

Proof (i):

Case (i) Let $k_1, k_2$ be two non-negative real numbers

$$k_1 \tilde{A} \oplus k_2 \tilde{B} = (k_1 a_1 + k_2 a_2, k_1 b_1 + k_2 b_2, k_1 c_1 + k_2 c_2, k_1 d_1 + k_2 d_2)$$

$$R\left( k_1 \tilde{A} \oplus k_2 \tilde{B} \right) = \frac{1}{2} (k_1 c_1 + k_2 c_2 + k_1 b_1 + k_2 b_2) \frac{1}{6} \left[ \begin{array}{c} (3(k_1 c_1 + k_2 c_2) - 2(k_1 a_1 + k_2 a_2) - (k_1 b_1 + k_2 b_2)) \\ (k_1 c_1 + k_2 c_2 - 3(k_1 b_1 + k_2 b_2)) + 2(k_1 d_1 + k_2 d_2) \\ 2(k_1 c_1 + k_2 c_2 + k_1 b_1 + k_2 b_2) \end{array} \right]$$
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\[ k \frac{1}{2} (c_1 + b_1) \frac{1}{6} ((3c_1 - 2a_1 - b_1)(c_1 - 3b_1 + 2d_1) + 2) + k_2 \frac{1}{2} (c_2 + b_2) \frac{1}{6} ((3c_2 - 2a_2 - b_2)(c_2 - 3b_2 + 2d_2) + 2) \]

\[ = k_1 R(\tilde{A}) \oplus k_2 R(\tilde{B}) \]

Similarly the result can be proved for case (ii) and case (iii).

**Proof (ii):** Let \( \tilde{A} = (a, b, c, d) \Rightarrow -\tilde{A} = (-d, -c, -b, -a) \)

\[ R(-\tilde{A}) = \left( \frac{-c - b}{2} \left( \frac{3c + 2a + b}{6} \right) \right) \]

\[ = -\left( \frac{c + b}{2} \left( \frac{3c - 2a - b}{6} \right) \right) \]

\[ R(-\tilde{A}) = -R(\tilde{A}) \]

**Proof (iii):** \( R(\tilde{A}) \oplus (-\tilde{A}) = R(\tilde{A}) \oplus R(\tilde{A}) \) (by (i))

\[ = R(\tilde{A}) \oplus R(\tilde{A}) \) (by (ii))

\[ = 0. \]

5. Numerical Examples

In this section, the proposed method is first explained by ranking some fuzzy numbers.

**Example 5.1**

Let \( \tilde{A} = (3, 5, 7; 1) \) and \( \tilde{B} = \left( 4, \frac{51}{8}, 1 \right) \) be triangular fuzzy numbers

Then \( O_\tilde{A}(\overline{x}_0, \overline{y}_0) = (5.16666) \ O_\tilde{B}(\overline{x}_0, \overline{y}_0) = (5.079) \)

Therefore, \( R(\tilde{A}) = 8.3333 \) and \( R(\tilde{B}) = 3.95 \)

Since \( R(\tilde{A}) > R(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B} \).

**Example 5.2**

Let \( \tilde{A} = (0, 1.2; 1) \) and \( \tilde{B} = \left( \frac{1}{5}, \frac{1}{4}, 1 \right) \) be triangular fuzzy numbers

Then \( O_{\tilde{A}}(\overline{x}_0, \overline{y}_0) = (1.06666) \) and \( O_{\tilde{B}}(\overline{x}_0, \overline{y}_0) = (1.05333) \)

Therefore, \( R(\tilde{A}) = 0.66666 \) and \( R(\tilde{B}) = 0.5333 \)

Since \( R(\tilde{A}) > R(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B} \)

**Example 5.3**

Let \( \tilde{A} = (0, 1, 2; 1) \Rightarrow -\tilde{A} = (-2, -1, 0; 1) \) and \( \tilde{B} = \left( \frac{1}{5}, \frac{7}{4}, 1 \right) \Rightarrow -\tilde{B} = \left( -\frac{7}{4}, -1, -\frac{1}{5}; 1 \right) \)
Step 1: \( O_A(\overline{x}_0, \overline{y}_0) = (-1, 0.6666) \) and \( O_B(\overline{x}_0, \overline{y}_0) = (-1, 0.5333) \)

Therefore, \( R(-\overline{A}) = 0.6666 \) and \( R(-\overline{B}) = 0.5333 \)

Since \( R(-\overline{A}) < R(-\overline{B}) \Rightarrow -\overline{A} < -\overline{B} \)

From examples 5.2 and 5.3 we see that the proposed method can rank fuzzy numbers and their images as it is proved that \( \overline{A} > \overline{B} \Rightarrow -\overline{A} < -\overline{B} \).

**Example 5.4**

Let \( \overline{A} = (0.1, 0.3, 0.5; 1) \) and \( \overline{B} = (0.2, 0.3, 0.4; 1) \)

Then \( O_A(\overline{x}_0, \overline{y}_0) = (0.3, 0.3466) \) and \( O_B(\overline{x}_0, \overline{y}_0) = (0.3, 0.3366) \)

Therefore, \( R(\overline{A}) = 0.1039 \) and \( R(\overline{B}) = 0.10099 \)

Since \( R(\overline{A}) > R(\overline{B}) \Rightarrow \overline{A} > \overline{B} \)

**Example 5.5**

Let \( \overline{A} = (0.1, 0.3, 0.5; 0.8) \) and \( \overline{B} = (0.1, 0.3, 0.5; 1) \) be non-normal triangular fuzzy numbers.

Then \( O_A(\overline{x}_0, \overline{y}_0) = (0.24, 0.28) \) and \( O_B(\overline{x}_0, \overline{y}_0) = (0.3, 0.346) \)

Therefore, \( R(\overline{A}) = 0.0672 \) and \( R(\overline{B}) = 0.1038 \)

Since \( R(\overline{A}) < R(\overline{B}) \Rightarrow \overline{A} < \overline{B} \)

From example 5.5 it is clear that the proposed method can rank fuzzy numbers with different height and same spreads.

**Example 5.6**

Let \( \overline{A} = (0.1, 0.2, 0.4, 0.5; 1) \) and \( \overline{B} = (0.1, 0.3, 0.5; 1) \)

Then \( O_A(\overline{x}_0, \overline{y}_0) = (0.3, 0.44) \) and \( O_B(\overline{x}_0, \overline{y}_0) = (0.3, 0.3466) \)

Therefore, \( R(\overline{A}) = 0.132 \) and \( R(\overline{B}) = 0.10398 \)

Since \( R(\overline{A}) > R(\overline{B}) \Rightarrow \overline{A} > \overline{B} \)

All the above examples are proved using step 1 only.

**Example 5.7**

Let \( \overline{A} = (0.1, 0.3, 0.5; 1) \) and \( \overline{B} = (0.2, 0.3, 0.7; 1) \)

**Step 1:** \( O_A(\overline{x}_0, \overline{y}_0) = (0.3, 0.3466) \) and \( O_B(\overline{x}_0, \overline{y}_0) = (0.3, 0.3466) \)

Therefore, \( R(\overline{A}) = 0.1039 \) and \( R(\overline{B}) = 0.1039 \)

Since \( R(\overline{A}) = R(\overline{B}) \), So goto step 2.

**Step 2:** \( m(\overline{A}) = 0.3 \) and \( m(\overline{B}) = 0.3 \)

Since \( m(\overline{A}) = m(\overline{B}) \), So goto step 3.

**Step 3:**

\( s(\overline{A}) = 0.4 \) and \( s(\overline{B}) = 0.4 \)

Since \( s(\overline{A}) = s(\overline{B}) \), So go to step 4.

**Step 4:**


6. Results and discussion

In this section the advantages of the proposed method is shown by comparing with other existing methods in literature, where the methods cannot discriminate fuzzy numbers. The results are shown in Table I and Table II.

Example 6.1
Consider two fuzzy numbers \( \tilde{A} = (1, 4, 5) \) and \( \tilde{B} = (2, 3, 6) \)

By Liou and Wang Method [17], it is clear that the two fuzzy numbers are equal for all the decision makers as

\[
I_l^\alpha(\tilde{A}) = 4.5\alpha + (1 - \alpha)2.5 \quad \text{and} \quad I_l^\alpha(\tilde{B}) = 4.5\alpha + (1 - \alpha)2.5
\]

Which is not even true by intuition.

By using our method we have

\[
O_A(x_0, y_0) = (4.13333) \quad \text{and} \quad O_B(x_0, y_0) = (3.13333)
\]

Therefore, \( R(\tilde{A}) = 5.3332 \) and \( R(\tilde{B}) = 3.9999 \) \( \Rightarrow \tilde{A} > \tilde{B} \)

Since \( R(\tilde{A}) > R(\tilde{B}) \) \( \Rightarrow \tilde{A} > \tilde{B} \)

Example 6.2
Let \( \tilde{A} = (0.1, 0.3, 0.5; 1) \), \( \tilde{B} = (1, 1, 1, 1) \)

Cheng [7] ranked fuzzy numbers with the distance method using the Euclidean distance between the Centroid point and original point. Where as Chu and Tsao [9] proposed a ranking function which is the area between the centroid point and original point. Their centroid formulae are given by

\[
(\bar{x}_0, \bar{y}_0) = \left(\frac{w(d^2 - 2e^2 + 2f^2 - a^2 + dc - ab) + 3(c^2 - b^2)}{3w(d - c + b - a) + 6(c - b)}\right) \cdot \left(1 + \frac{(b + c) - (a + d)(1 - w)}{(b + c - a - d) + 2(a + d)w}\right)
\]

Both these centroid formulae cannot rank crisp numbers which are a special case of fuzzy numbers as it can be seen from the above formulae that the denominator in the first coordinate of their centroid formulae is zero, and hence centroid of crisp numbers are undefined for their formulae. By using our method, we have
Therefore, \( R(\tilde{A}) = 0.10398 \) and \( R(\tilde{B}) = 1.3333 \)

Since \( R(\tilde{A}) < R(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B} \)

From this example it is proved that the proposed method can rank crisp numbers whereas, other methods failed to do so.

**Example 6.3**

Consider four fuzzy numbers
\[
\tilde{A}_1 = (0.1, 0.2, 0.3; 1), \tilde{A}_2 = (0.2, 0.5, 0.8; 1), \tilde{A}_3 = (0.3, 0.4, 0.9; 1), \tilde{A}_4 = (0.6, 0.7, 0.8; 1)
\]

Which were ranked earlier by Yager[22], Fortemps and Roubens[11], Liou and Wang[17], and Chen [5] as shown in Table I.

<table>
<thead>
<tr>
<th>Fuzzy ranking</th>
<th>( \tilde{A}_1 )</th>
<th>( \tilde{A}_2 )</th>
<th>( \tilde{A}_3 )</th>
<th>( \tilde{A}_4 )</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yager [22]</td>
<td>0.20</td>
<td>0.50</td>
<td>0.50</td>
<td>0.70</td>
<td>( \tilde{A}_4 &gt; \tilde{A}_2 \approx \tilde{A}_3 &gt; \tilde{A}_1 )</td>
</tr>
<tr>
<td>Fortemps &amp; Roubens[11]</td>
<td>0.20</td>
<td>0.50</td>
<td>0.50</td>
<td>0.70</td>
<td>( \tilde{A}_4 &gt; \tilde{A}_2 \approx \tilde{A}_3 &gt; \tilde{A}_1 )</td>
</tr>
<tr>
<td>Liou &amp; Wang[17]</td>
<td>( \alpha = 1 )</td>
<td>0.25</td>
<td>0.65</td>
<td>0.65</td>
<td>( \tilde{A}_4 &gt; \tilde{A}_2 \approx \tilde{A}_3 &gt; \tilde{A}_1 )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0.5 )</td>
<td>0.20</td>
<td>0.50</td>
<td>0.50</td>
<td>( \tilde{A}_4 &gt; \tilde{A}_2 \approx \tilde{A}_3 &gt; \tilde{A}_1 )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0 )</td>
<td>0.15</td>
<td>0.35</td>
<td>0.35</td>
<td>( \tilde{A}_4 \approx \tilde{A}_3 \approx \tilde{A}_1 &gt; \tilde{A}_2 )</td>
</tr>
<tr>
<td>Chen [5]</td>
<td>( \beta = 1 )</td>
<td>-0.20</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.20</td>
</tr>
<tr>
<td></td>
<td>( \beta = 0.5 )</td>
<td>-0.20</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.20</td>
</tr>
<tr>
<td></td>
<td>( \beta = 0 )</td>
<td>-0.20</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.20</td>
</tr>
<tr>
<td>Proposed</td>
<td>0.0673</td>
<td>0.0726</td>
<td>0.2</td>
<td>0.2356</td>
<td>( \tilde{A}_4 &gt; \tilde{A}_2 \approx \tilde{A}_3 &gt; \tilde{A}_1 )</td>
</tr>
</tbody>
</table>

It can be seen from Table I that none of the methods discriminates fuzzy numbers. Yager[22] and Fortemps and Roubens [11] methods failed to discriminate the fuzzy numbers \( \tilde{A}_2 \) and \( \tilde{A}_3 \), Whereas the methods of Liou and Wang[17], and Chen [5] cannot discriminate the fuzzy numbers \( \tilde{A}_2 \), \( \tilde{A}_3 \) and \( \tilde{A}_4 \).

By using our method, we have
Ordering generalized trapezoidal fuzzy numbers

Therefore, 
\[ R(\tilde{A}_1) = 0.0673, \quad R(\tilde{A}_2) = 0.18165, \quad R(\tilde{A}_3) = 0.14, \quad R(\tilde{A}_4) = 0.2356 \Rightarrow \tilde{A}_4 > \tilde{A}_2 > \tilde{A}_3 > \tilde{A}_1 \]

Example 6.4
In this we consider seven sets of fuzzy numbers available in literature and the comparative study is presented in Table II.

| Set 1: & \tilde{A} = (0.2,0.4,0.6,0.8;0.35) and \tilde{B} = (0.1,0.2,0.3,0.4;0.7) |
| Set 2: & \tilde{A} = (0.1,0.2,0.4,0.5;1) and \tilde{B} = (0.1,0.3,0.3,0.5;1) |
| Set 3: & \tilde{A} = (0.1,0.2,0.4,0.5;1) and \tilde{B} = (1,1,1,1;1) |
| Set 4: & \tilde{A} = (0.1,0.2,0.4,0.5;1) and \tilde{B} = (0.1,0.3,0.3,0.5;1) |
| Set 5: & \tilde{A} = (0.3,0.5,0.5,1;1), and \tilde{B} = (0.1,0.6,0.6,0.8;1) |
| Set 6: & \tilde{A} = (0.0,0.4,0.6,0.8,1), \tilde{B} = (0.2,0.5,0.5,0.9,1) and \tilde{C} = (0.1,0.6,0.7,0.8,1), |
| Set 7: & \tilde{A} = (0.1,0.2,0.4,0.5,1;1), and \tilde{B} = (0,2,0.2,1;) |

Table II. A Comparison of proposed fuzzy ranking with existing ranking methods

<table>
<thead>
<tr>
<th>Methods</th>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
<th>Set 4</th>
<th>Set 5</th>
<th>Set 6</th>
<th>Set 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cheng [7]</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} \approx \tilde{B}</td>
<td>***</td>
<td>\tilde{A} \approx \tilde{B}</td>
<td>\tilde{A} &gt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B} &lt; \tilde{C}</td>
<td>***</td>
</tr>
<tr>
<td>Chu and Tsao [9]</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} \approx \tilde{B}</td>
<td>***</td>
<td>\tilde{A} \approx \tilde{B}</td>
<td>\tilde{A} &gt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B} &lt; \tilde{C}</td>
<td>***</td>
</tr>
<tr>
<td>Chen and Chen [4]</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{C} &lt; \tilde{B}</td>
<td>\tilde{A} &gt; \tilde{B}</td>
<td></td>
</tr>
<tr>
<td>Abbasbandy and Hajjari [1]</td>
<td>***</td>
<td>\tilde{A} \approx \tilde{B}</td>
<td>\tilde{A} \approx \tilde{B}</td>
<td>\tilde{A} \approx \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B} &lt; \tilde{C}</td>
<td>\tilde{A} &gt; \tilde{B}</td>
<td></td>
</tr>
<tr>
<td>Chen and Chen [6]</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B} &lt; \tilde{C}</td>
<td>\tilde{A} &gt; \tilde{B}</td>
<td></td>
</tr>
<tr>
<td>Kumar et al. [16]</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B} &lt; \tilde{C}</td>
<td>\tilde{A} &gt; \tilde{B}</td>
<td></td>
</tr>
<tr>
<td>Proposed method</td>
<td>\tilde{A} &gt; \tilde{B}</td>
<td>\tilde{A} &gt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{A} &lt; \tilde{B}</td>
<td>\tilde{B} &lt; \tilde{A} &lt; \tilde{C}</td>
<td>\tilde{A} &gt; \tilde{B}</td>
<td></td>
</tr>
</tbody>
</table>

*** not comparable

7. Conclusions and future work

This paper recommends a method that ranks fuzzy numbers which is simple and concrete. This method ranks trapezoidal as well as triangular fuzzy numbers and their images. This method also ranks crisp numbers which are special case of fuzzy numbers whereas some methods proposed in literature cannot rank crisp numbers. This method which is simple and easier in calculation not only gives...
satisfactory results to well defined problems, but also gives a correct ranking order to problems. Comparative examples are used to illustrate the advantages of the proposed method. Application of this ranking procedure in various fuzzy decision making problems, fuzzy risk analysis, fuzzy optimization problems and fuzzy network analysis, fuzzy transportation problems etc. is left as future research work.

References


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