Left Quasi-Noetherian Modules

Falih A. M. Aldosary and Amani M. A. Alfadli

Department of Mathematic, Umm Al-Qura University
Makkah P.O. Box 56199, Saudi Arabia
fadosary@uqu.edu.sa

Abstract

In this paper, we study a new class of left Quasi-Noetherian Rings and Modules. Which is a generalization of nilpotent rings and left Noetherian Modules.

Keywords: Quasi-Noetherian Modules, primary decomposition and Goldie's rings.

Introduction

Let $R$ be a ring without identity. A left $R$-Module $M$ is said to be left Quasi-Noetherian if for every ascending chain $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ of $R$-submodules of $M$, there exists $m \in \mathbb{Z^+}$ such that $R^m(U_n, N_n) \subseteq N_m$. It is clear that any left Noetherian $R$-module is left Quasi-Noetherian, also if $RM=0$, then $M$ is left Quasi-Noetherian module. Note that if $M$ is a unital left Quasi-Noetherian module, then $M$ is left Noetherian. we say that the ring $R$ is left Quasi-Noetherian ring if $R^R$ is Quasi-Noetherian. Note that any nilpotent ring is left Quasi-Noetherian, however $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & 0 \end{bmatrix}$ is a non-nilpotent left Quasi-Noetherian but not Noetherian. Also $R = \begin{bmatrix} 2\mathbb{Z} \\ 0 \\ \mathbb{Z}_2[x_1, x_2, \ldots] \end{bmatrix}$ is Quasi-Noetherian but not Noetherian. It is clear that if $R = (2\mathbb{Z}, +, \cdot)$, then $R$ and $R[x_1], R[x_1, x_2], \ldots, R[x_1, \ldots, x_n]$ are Quasi-Noetherian. Note that if $\mathbb{R}$ is the field of real numbers and $R = \mathbb{R}[x]$, then $\langle x \rangle = x\mathbb{R}[x]$ is Quasi-Noetherian domain which is not Noetherian.
First we study the problem of finding conditions which are equivalent to the definition of a left Quasi-Noetherian module (Theorem 1.1). Next we study the relation between left Noetherian and left Quasi-Noetherian modules, in particular we show that if $RM$ is left Noetherian, then $M$ is left Quasi-Noetherian (Theorem 1.2). Then we show that the class of a left Quasi-Noetherian module is S-closed, Q-closed (Proposition 1.3), E-closed (Proposition 1.5) and as a consequence of this result we show that a finite direct sum of left Quasi-Noetherian Modules is left Quasi-Noetherian (Corollary 1.6). Finally we will describe the class of left Quasi-Noetherian rings, we show the following: the ring $R$ is left Quasi-Noetherian if and only if $N = N(R)$ is nilpotent and each of the $R$-modules $R/N, N/N^2, N^2/N^3, \ldots$ is left Quasi-Noetherian (Theorem 1.7).

In section two we study the relationship between left Quasi-Noetherian rings and some certain special types of rings. First we will give its relationship with left Goldie rings, in particular we show that if $R$ is a left Quasi-Noetherian ring and $r(R) = 0$, then $R$ is a left Goldie ring (Theorem 2.2). Then we prove (Theorem 2.4) Let $R$ be a left Quasi-Noetherian ring .(a) If $Q(R)$ is a ring of left quotient of $R$ then $Q(R)$ is left Noetherian, (b) If $R$ without zero divisors then it has a left quotient ring which is a division ring. Finally we prove (Theorem 2.5 and Theorem 2.9) if $R$ is a left Quasi-Noetherian ring without zero divisors then the polynomial ring $R[x]$ and the ring of formal power series $R[[x]]$ are left Quasi-Noetherian.

In section three we study the ideal structures in Quasi-Noetherian rings. In particular we show that if $R$ is a commutative Quasi-Noetherian ring, then every ideal in $R$ has a primary decomposition (Theorem 3.).Then we study the existence of prime ideals in the left Quasi-Noetherian rings, we show that If $R$ is a non-nilpotent left Quasi Noetherian ring. Then there exist a prime ideal in $R$ which is a right annihilator of some non-zero ideal of $R$ (Theorem 3.6). Finally we prove (Theorem 3.7) which asserts that every ideal in left Quasi-Noetherian ring contains a product of prime ideals.

1. Definitions and basic properties

Let $M$ be a left $R$-module, $N$ is a submodule of $M$, then \( N: M = \{ x \in M : Rx \subseteq N \} \) is a submodule of $M$, which is called the quotient of $N$ by $R$.

Now we prove the following

**Theorem 1.1:**

Let $M$ be a left $R$-module. Then the following conditions are equivalent:

(a) $M$ is left Quasi-Noetherian.
(b) In each non-empty collection \( \Gamma \) of \( R \)-submodules of \( M \), there is an element \( N \) and an \( m \in \mathbb{Z}^+ \) such that \( R^m K \subseteq N \) for every \( K \in \Gamma \) and \( K \supseteq N \).

(c) In each non-empty collection \( \Gamma \) of \( R \)-submodules of \( M \) such that \( K \in \Gamma \)

\[ \Rightarrow K :_M R = \{ m \in M \mid Rm \subseteq K \} \in \Gamma \], there exists a maximal element.

(d) For every ascending chain of \( R \)-submodules \( N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots \) there exists \( m \in \mathbb{Z}^+ \) such that the ascending chain \( N_1 :_M R^m \subseteq N_2 :_M R^m \subseteq \cdots \subseteq N_n :_M R^m \subseteq \cdots \) of \( R \)-submodules terminates.

**Proof:**

(a) \( \Rightarrow \) (b) Let \( M \) be a left Quasi-Noetherian module. Let \( \Gamma \) be a non-empty collection of \( R \)-submodules of \( M \). Suppose that for all \( N \in \Gamma \) and \( m \in \mathbb{Z}^+ \), there exists \( K \in \Gamma \) such that \( K \supseteq N \), but \( R^m K \not\subseteq N \). Pick an \( R \)-submodule \( N_1 \in \Gamma \) then there exists \( N_2 \in \Gamma \), \( N_1 \subseteq N_2 \) such that \( RN_2 \not\subseteq N_2 \) likewise there exists \( N_2 \in \Gamma \), \( N_2 \subseteq N_3 \) such that \( R^2 N_3 \not\subseteq N_2 \), continuing in this way, we obtain an ascending chain \( N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots \) of \( R \)-submodules of \( M \) such that \( R^n N_{n+1} \not\subseteq N_n \), for all \( n \in \mathbb{Z}^+ \). Therefore \( R^m (\bigcup_n N_n) \not\subseteq N_m \) for all \( m \in \mathbb{Z}^+ \) which contradicts the hypothesis that \( M \) is a left Quasi-Noetherian module.

(b) \( \Rightarrow \) (c) Let \( \Gamma \) be a non-empty collection of \( R \)-submodules of \( M \) such that \( K :_M R \in \Gamma \) for all \( K \in \Gamma \). Then \( K :_M R^n \in \Gamma \) for all \( n \in \mathbb{Z}^+ \). But \( K \subseteq K :_M R^n \) for all \( n \in \mathbb{Z}^+ \), hence by (b) there exists an \( m \in \mathbb{Z}^+ \) such that \( R^m (K :_M R^n) \subseteq K \) for all \( n \in \mathbb{Z}^+ \). Therefore \( K :_M R^n \subseteq K :_M R^m \) for all \( n \in \mathbb{Z}^+ \). Hence \( K :_M R^n = K :_M R^m \) for all \( n \geq m \), and \( \Gamma \) has a maximal element.

(c) \( \Rightarrow \) (d) Let \( N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots \) be any ascending chain of \( R \)-submodules and suppose that the ascending chain \( N_1 :_M R^m \subseteq N_2 :_M R^m \subseteq \cdots \subseteq N_n :_M R^m \subseteq \cdots \) of \( R \)-submodules does not terminate for all \( m \in \mathbb{Z}^+ \). Therefore the collection \( \Gamma = \{ N_1, \ldots, N_1 :_M R, \ldots, N_1 :_M R^m, \ldots \} \) is a non-empty collection of \( R \)-submodules and for all \( N \in \Gamma \) we have \( N :_M R \in \Gamma \) but \( \Gamma \) has no maximal element, which is a contradiction.

(d) \( \Rightarrow \) (a) Let \( N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots \) be any ascending chain of \( R \)-submodules of \( M \) then there exists \( m \in \mathbb{Z}^+ \) such that \( N_1 :_M R^m \subseteq N_2 :_M R^m \subseteq \cdots \subseteq N_n :_M R^m \subseteq \cdots \) forms an ascending chain of \( R \)-submodules of \( M \) and by (d) there exists \( s \in \mathbb{Z}^+ \) such that \( N_s :_M R^m = N_n :_M R^m \) for all \( n \geq s \) then \( N_s :_M R^m = \bigcup_n (N_n :_M R^m) \) so \( R^m (\bigcup_n (N_n :_M R^m)) \subseteq N_s \) then \( R^m (\bigcup_n (N_n)) \subseteq N_s \). Take \( t = \max \{ m, s \} \) then \( R^t (\bigcup_n (N_n)) \subseteq N_t \), hence \( M \) is a left Quasi-Noetherian module.

\( \square \)
Theorem 1.2:
Let $M$ be a left $R$-module. If $RM$ is left Noetherian, then $M$ is left Quasi-Noetherian.

Proof:
Let $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ be any ascending chain of $R$-submodules of $M$. Then $RA_1 \subseteq RA_2 \subseteq \ldots \subseteq RA_n \subseteq \ldots$ is an ascending chain of $R$-submodules of $RM$. But $RM$ is left Noetherian, hence $RA_r = RA_s \subseteq A_s$ for all $r \geq s$. Therefore $R(\bigcup A_n) \subseteq A_s$ and $M$ is left Quasi-Noetherian. □

Note that the converse of Theorem (1.2) is not necessarily true as the following example shows: Let $R = \{ \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} : r \in \mathbb{Z}, k \in \mathbb{Z}^+, p \text{ is prime} \}$ and $M = \{ \begin{pmatrix} a \\ b \end{pmatrix} : a \in \mathbb{Z}, b \in \mathbb{Q} \}$ so $M$ is a left $R$-module, and for every ascending chain $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ of $R$-submodules of $M$ there exists $m = 2 \in \mathbb{Z}^+$ such that $R^m(U_n N_n) = R^2(U_n N_n) = 0 \subseteq N_2$. Hence $M$ is a left Quasi-Noetherian module. But $RM = \{ \begin{pmatrix} 0 \\ t & 0 \end{pmatrix} : t \in \mathbb{Z}, k \in \mathbb{Z}^+, p \text{ is prime} \}$ is $R$, is not left Noetherian.

Now let $\mathcal{M}$ be a class of modules. Then we say that $\mathcal{M}$ is S-closed if $N$ is a submodule of $M$ and $M \in \mathcal{M}$ then $N \in \mathcal{M}$. We say that $\mathcal{M}$ is Q-closed if $M \in \mathcal{M}$ and $N$ is a submodule of $M$ then $M/N \in \mathcal{M}$. We say that $\mathcal{M}$ is E-closed if $N$ is a submodule of $M$ and $N,M/N \in \mathcal{M}$ then $M \in \mathcal{M}$.

Note that if $\mathcal{M}$ is the class of left Quasi-Noetherian Modules, and $M \in \mathcal{M}$, $N$ is a submodule of $M$, then since any submodule of $N$ is a submodule of $M$ it follows that any ascending chain of submodules of $N$ is an ascending chain of submodules of $M$. Therefore the chain terminates and $N \in \mathcal{M}$. Hence $\mathcal{M}$ is S-closed.

Now we prove the following

Proposition 1.3

Let $\mathcal{M}$ be the class of left Quasi-Noetherian Modules. Then $\mathcal{M}$ is Q-closed.

Proof:
Suppose that $M \in \mathcal{M}$ and let $N$ be an $R$-submodule of $M$ and let $\pi : M \to M/N = \bar{M}$ be the natural homomorphism. Now if $\bar{N}_1 \subseteq \bar{N}_2 \subseteq \ldots \subseteq \bar{N}_n \subseteq \ldots$ be an ascending chain of $R$-submodules of $\bar{M}$. Put $N_k = \pi^{-1}(\bar{N}_k)$, $k = 1,2,\ldots$ Then $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ is an ascending chain of $R$-submodules of $M$, but $M$ is left Quasi-Noetherian, hence there exists an $m \in \mathbb{Z}^+$ such that
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Let submodules of Corollary 1.6 be the class of left Quasi-Noetherian modules. If $M/N \in \mathcal{M}$ for all $0 \neq N \leq M$, then $M \in \mathcal{M}$.

Proof:
Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ be any ascending chain of $R$-submodules of $M$. Then $A_2/A_1 \subseteq A_3/A_1 \subseteq \cdots \subseteq A_n/A_1 \subseteq \cdots$ is an ascending chain of $R$-submodules of $M/A_1$. But $M/A_1$ is left Quasi-Noetherian module, hence there exists $m \in \mathbb{Z}^+$ such that $R^m(U_n A_n/A_1) \subseteq A_m/A_1$. Therefore $R^m(U_n A_n) \subseteq A_m$. Hence $M$ is a left Quasi-Noetherian module.

Proposition 1.5

Let $\mathcal{M}$ be the class of left Quasi-Noetherian Modules. Then $\mathcal{M}$ is E-closed.

Proof:
Let $N$ be an $R$-submodule of $M$, and suppose that $N$ and $M/N$ in $\mathcal{M}$. Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be any ascending chain of $R$-submodules of $M$. Then $N_1 \cap N \subseteq N_2 \cap N \subseteq \cdots \subseteq N_n \cap N \subseteq \cdots$ is an ascending chain of $R$-submodules of $N$ and $N_1/N \subseteq N_2/N \subseteq \cdots \subseteq N_n/N \subseteq \cdots$ is an ascending chain of $R$-submodules of $M/N$. But $N$ and $M/N$ in $\mathcal{M}$, hence there exists $t \in \mathbb{Z}^+$ such that $R^t(U_n N_n \cap N) \subseteq N_t \cap N$, which implies that $R^t((U_n N_n) \cap N) \subseteq N_t \cap N$. Also there exists $s \in \mathbb{Z}^+$ such that $R^s(U_n N_n + N) \subseteq N_s + N$, which implies $R^s(U_n N_n + N) \subseteq N_s + N$. Take $m = \max \{t,s\}$, then $R^m(U_n N_n \cap N) \subseteq N_m \cap N$ (1) and $R^m(U_n N_n + N) \subseteq N_m + N$ (2).

Now $R^m(U_n N_n) \subseteq R^m(U_n N_n + N) \cap (U_n N_n) \subseteq (N_m + N) \cap (U_n N_n)$, and $R^m(U_n N_n + N) \subseteq (N_m \cap N) + N_m$ by modular law so $R^m(U_n N_n) \subseteq R^m((U_n N_n \cap N) + N_m) = R^m(U_n N_n \cap N) + R^m N_m \subseteq (N_m \cap N) + N_m = N_m$. Therefore $R^{2m}(U_n N_n) \subseteq N_m \subseteq N_{2m}$. Hence $M$ is a left Quasi-Noetherian module.

An immediate consequence of Theorem 1.5, we have the following

Corollary 1.6

A finite direct sum of left Quasi-Noetherian modules is left Quasi-Noetherian.
By the nil radical $N = N(R)$ of a ring $R$ we mean the sum of all nilpotent ideals of $R$, which is a nil ideal. It is well known [7, P.28, Theorem 2], that $N$ is the sum of all nilpotent left ideals of $R$ and it is the sum of all nilpotent right ideals of $R$.

Now we regarded $R$ as $R$-module, so we have the following, which describe the class of left Quasi-Noetherian rings.

**Theorem 1.7:**

A ring $R$ is left Quasi-Noetherian if and only if $N = N(R)$ is nilpotent and each of the $R$-modules $R/N, N/N^2, N^2/N^3, \ldots$ is left Quasi-Noetherian.

To prove this we need the following.

**Lemma 1.8:**

Let $R$ be a left Quasi-Noetherian ring. Then the nil radical $N$ of $R$ is nilpotent.

**Proof:**

Let $F$ be the family of all nilpotent left ideals of $R$. Then $0 \in F$ and $F \neq \emptyset$. By Theorem 1.1 there is a nilpotent left ideal $I$ and $m \in \mathbb{Z}^+$, such that $R^m J \subseteq I$ for any nilpotent left ideal $J \supseteq I$. If $I^n = 0$, then $J^{mn+n} \subseteq (R^m J)^n \subseteq I^n = 0$ and $J^{mn+n} = 0$ for every nilpotent left ideal $J \supseteq I$. Now let $a_1, \ldots, a_{mn+n} \in N$. Since the sum of two nilpotent left ideals is nilpotent, it follows that $a_i \in J_i$ for some nilpotent left ideal $J_i, i = 1, \ldots, mn + n$, so if $J = \sum_{i=1}^{mn+n} J_i + I$, then $J$ is nilpotent and $J \supseteq I$. But $J^{mn+n} = 0$, hence $a_1 \ldots a_{mn+n} = 0$ therefore $N^{mn+n} = 0$. □

As the consequence of Lemma 1.8 we have the following

**Corollary 1.9:**

Let $R$ be a left Quasi-Noetherian ring. Then every nil left ideal in $R$ is nilpotent.

**Proof of Theorem 1.7:**

Suppose that $R$ is a left Quasi-Noetherian ring. Then by Lemma 1.8 $N$ is nilpotent. Now $R M = R R \in \mathcal{M}$, and $N^i \lhd R$ for all $i$, therefore $N^i \leq M$ for all $i$. But $\mathcal{M}$ is Q-closed, hence $R/N^i$ is a left Quasi-Noetherian $R$-module for all $i$. Also $N^i/N^{i+1}$ is an $R$-submodule of $R/N^{i+1}$ for all $i \geq 1$ so each $N^i/N^{i+1}$ is a left Quasi-Noetherian $R$-module.

Conversely: Since $R/N \cong R/N^2/N/N^2$ and $R/N$, $N/N^2$ are left Quasi- Noetherian $R$-module, it follows from Theorem (\mathcal{M} is an E-closed), that $R/N^2 \in \mathcal{M}$, and by induction, $R/N^i \in \mathcal{M}$ for all $i$. But $N$ is nilpotent, hence there exists $m \in \mathbb{Z}^+$ such
that $N^m = 0$. Therefore $R \cong R/N^m$ is a left Quasi-Noetherian $R$-module. Hence $R$ is a left Quasi-Noetherian ring.

\[ \square \]

2. Goldie's Rings, Rings of left Quotient and Polynomial Rings.

In this section we will study the relationship between left Quasi-Noetherian rings and some certain special types of rings (most notably Goldie rings, rings of left quotients and polynomial rings).

If $R$ be a ring $I \triangleleft R$. Then $0_r I = r(I) = \{ x \in R : Ix = 0 \}$ is called the right annihilator of $I$ in $R$. The left annihilator $l(I)$ can be defined similarly. We say that $I$ is an annihilator ideal if $I = r(l(I))$.

A ring $R$ is said to be a left Goldie ring if:

(a) $R$ satisfies the a.c.c on left annihilator ideals.

(b) $R$ has no infinite direct sum on left ideals.

Not that condition (b) is equivalent to that $R$ satisfies the a.c.c on complement left ideals. The right Goldie ring defined similarly.

Let $R$ be a ring. Then:

(a) An element $x$ of $R$ is said to be regular if its neither a left nor right zero divisor in $R$ i.e. $r(x) = \ell(x) = 0$.

(b) The ring of left quotients of $R$ is a ring $Q(R)$ which satisfies the following conditions:

(1) $R$ is a subring of $Q(R)$.

(2) Each regular element of $R$ is a unit of $Q(R)$.

(3) Each element $q$ of $Q(R)$ is of the form $c^{-1}a$ for some elements $a$ and $c$ of $R$ with $c$ regular.

**Remark 2.1**

If $R$ is a left Quasi-Noetherian ring, then $R$ need not be a left Goldie ring as the following examples show:

(a) Let $R = \begin{bmatrix} 0 & 0 \\ Q & 0 \end{bmatrix}$ Then $R^2 = 0$, hence $R$ is a left Quasi-Noetherian ring, but it is not a left Goldie ring for if $I_n = \left\{ \begin{bmatrix} m \\ zn \end{bmatrix} : m \in \mathbb{Z}, n \in \mathbb{Z}^+ \right\}$, then $I_n$ is a left annihilator ideal in $R$ for all $n$, and $I_1 \subseteq I_2 \subseteq \cdots$ is an infinite strictly ascending chain of left annihilator ideals in $R$.

Note that $r(R) \neq 0$, the right annihilator of $R$. 


(b) Let $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & 0 \end{bmatrix}$. Then $R$ is a left Quasi-Noetherian ring but it's not a left Goldie ring for if $I_n = \begin{bmatrix} 0 & 0 \\ m_{2^n} & 0 \end{bmatrix} : m \in \mathbb{Z}, n \in \mathbb{Z^+}$, then $I_n$ is a left annihilator ideal in $R$ for all $n$, and $I_1 \subsetneq I_2 \subsetneq \cdots$ is an infinite strictly ascending chain of left annihilator ideals in $R$. Note that: $r(R) = \begin{bmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{bmatrix} \neq 0$, but $\ell(R) = 0$.

However we prove the following which gives the relationship between left Quasi-Noetherian rings and left Goldie rings.

**Theorem 2.2:**

Let $\mathfrak{X}$ be a class of left Quasi-Noetherian rings and $R \in \mathfrak{X}$. If $r(R) = 0$, then $R$ is a left Goldie ring.

**Proof:**

First we show that any ascending chain of left annihilator ideals terminates. Let $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$ be any ascending chain of left annihilator ideals of $R$. Suppose that $J_i = l(I_i)$ for all $i$. Since $R$ is a left Quasi-Noetherian ring then there exists $m \in \mathbb{Z^+}$ such that $R^m(U_n J_n) \subseteq J_m = l(I_m)$, therefore $R^m(U_n J_n)I_m = 0$, and $R(R^{m-1}(U_n J_n)I_m) = 0$. But $r(R) = 0$, hence $R^{m-1}(U_n J_n)I_m = 0$. Continuing in this way we have $R(U_n J_n)I_m = 0$ therefore $(U_n J_n)I_m = 0$, and $U_n J_n \subseteq l(I_m) = J_m$. Hence $J_m = J_{m+1} = \cdots$ and the chain terminates. Now let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be any ascending chain of complement left ideals of $R$. Since $R$ is a left Quasi-Noetherian ring then there exists $m \in \mathbb{Z^+}$ such that $R^m(U_n I_n) \subseteq I_m$. Now suppose that $I_m$ is a complement of $J_m$, then $(R^m(U_n I_n) \cap J_m) \subseteq I_m \cap J_m = 0$. But $(U_n I_n) \cap J_m \subseteq U_n I_n$ and $(U_n I_n) \cap J_m \subseteq J_m$, hence $R^m((U_n I_n) \cap J_m) \subseteq R^m(U_n I_n)$ and $R^m((U_n I_n) \cap J_m) \subseteq (U_n I_n) \cap J_m \subseteq J_m$. Therefore $R^m((U_n I_n) \cap J_m) \subseteq (R^m(U_n I_n)) \cap J_m = 0$ and $R(R^{m-1}((U_n I_n) \cap J_m)) = 0$. But $r(R) = 0$, hence $R^m((U_n I_n) \cap J_m) = 0$. Continuing in this way we have $(U_n I_n) \cap J_m = 0$, and by maximality of $I_m$, we have $U_n I_n = I_m$. Hence $I_m = I_{m+1} = \cdots$. Therefore $R$ is a left Goldie ring.

\[\square\]

**Corollary 2.3:**

(a) Let $R$ be a left Quasi-Noetherian ring with at least one regular element then $R$ is a left Goldie ring.

(b) Let $R$ be a semi-prime left Quasi-Noetherian ring. Then $R$ has a ring of left quotients which is a semi-simple left Artinian ring.
Proof:
(a) Since $R$ has at least one regular element it follows that $r(R) = 0$ and the result follows from Theorem 2.2.

(b) Since $R$ is a semi-prime, it follows that $r(R) = 0$, and by Theorem 2.2 $R$ is a semi-prime left Goldie ring. Therefore $R$ has a ring of left quotient which is semi-simple left Artinian (by Goldie's Theorem).

Now we prove the following which gives the relationship between a left Quasi-Noetherian ring and its ring of left quotient.

Theorem 2.4:
Let $R$ be a left Quasi-Noetherian ring.

(a) If $Q(R)$ is a ring of quotient of $R$ then $Q(R)$ is left Noetherian.

(b) If $R$ without zero divisors then it has a ring of left quotients which is a division ring.

Proof:
(a) Let $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$ be any ascending chain of left ideals of $Q(R)$. Then $J_1 \cap R \subseteq J_2 \cap R \subseteq \cdots \subseteq J_n \cap R \subseteq \cdots$ is an ascending chain of left ideals of $R$, so there exists $m \in \mathbb{Z}^+$ such that $R^m(U_n J_n \cap R) \subseteq J_m \cap R$. Now $(R)^{(R^m(U_n J_n \cap R))} \subseteq Q(R)(J_m \cap R) = J_m$. But $Q(R)R = Q(R)$. Hence $Q(R)R^m = Q(R)RR^{m-1} = Q(R)R^{m-1} = \cdots = Q(R)$. Therefore $(U_n J_n) = Q(R)(U_n J_n \cap R) \subseteq J_m$. Hence $J_m = J_{m+1} = \cdots$. Therefore $Q(R)$ is a left Noetherian ring.

(b) Since $R$ is a left Quasi-Noetherian ring without zero divisor it follows that $r(R) = 0$, hence $R$ is a left Goldie ring by Theorem 2.2. Therefore $R$ has a left ring of quotients which is a division ring [8, Lemma 4.1]

Note that the converse of Theorem 2.4 (a) is not necessarily true as the following example shows:
Let $R = F [x_1, x_2, \ldots]$ be a polynomial ring with infinitely many indeterminates over the field of characteristic zero $F$. Then its ring of quotient $Q(R) = K$, which is a field and it's Noetherian and so Quasi-Noetherian but $R$ is not.

Next we prove a small generalization of Hilbert's Basis Theorem, namely the following.

Theorem 2.5:
Let $R$ be a left Quasi-Noetherian ring without zero divisors. Then $R[x]$ is left Quasi-Noetherian.

To prove this we need the following.
Lemma 2.6:
Let \( R \) be a ring without zero divisors and \( J, K \) are left ideals in the polynomial ring \( R[x] \), \( J \subseteq K \) and \( t \geq 0 \). If \( L_t(J) = \{ a_t \in R | \exists a_0, a_1, ..., a_{t-1} \text{ such that } a_0 + a_1 x + \cdots + a_t x^t \in J \} \). Then
(a) \( L_t(J) \) is a left ideal in \( R \) and \( L_t(J) \subseteq L_{t+1}(J) \) for all \( t \geq 0 \).
(b) \( L_t(J) \subseteq L_t(K) \) for all \( t \geq 0 \).
(c) If there exists \( m \in \mathbb{Z}^+ \) such that \( R^m L_t(K) \subseteq L_t(J) \) for all \( t \geq 0 \), then \( (R^m[x])^m K \subseteq J \).

Proof: (a) See [9, p 418]

(b) Let \( a_t \in L_t(J) \) for all \( t \geq 0 \). Then there exists \( a_0, a_1, ..., a_{t-1} \) such that \( a_0 + a_1 x + \cdots + a_t x^t \in J \) but \( J \subseteq K \) so \( a_t \in L_t(K) \) for all \( t \geq 0 \), therefore \( L_t(J) \subseteq L_t(K) \) for all \( t \geq 0 \).

(c) Let \( f(x) \in (R[x])^m K \), \( \deg(f(x)) = t \geq 0 \). Then
\[
L_t(f(x)) \subseteq L_t((R[x])^m K) = L_t((R[x])^m L_{t-\tau}(K)) = L_{\tau}(R[x])L_{\tau_2}(R[x])...L_{\tau_m}(R[x])L_{t-\tau}(K), \sum_{i=1}^{m} r_i \subseteq R^m L_{t-\tau}(K) \subseteq L_{t-\tau}(J) \subseteq L_t(J).
\]
Therefore \( f(x) \in J \). Hence \( (R[x])^m K \subseteq J \).

Lemma 2.7: \( \bigcup_n L_t(I_n) = L_t(\bigcup_n I_n) \)

Proof: Let \( a_t \in \bigcup_n L_t(I_n) \). Then \( a_t \in L_t(I_n) \) for some \( n \), hence there exists \( a_0, ..., a_{t-1} \) such that \( a_0 + a_1 x + \cdots + a_t x^t \in I_n \) for some \( n \). Therefore \( a_0 + a_1 x + \cdots + a_t x^t \in \bigcup_n I_n \) and \( a_t \in L_t(\bigcup_n I_n) \). Hence \( \bigcup_n L_t(I_n) \subseteq L_t(\bigcup_n I_n) \)

Now let \( b_t \in L_t(\bigcup_n I_n) \). Then there exists \( b_0, ..., b_{t-1} \) such that \( b_0 + b_1 x + \cdots + b_t x^t \in \bigcup_n I_n \). Therefore \( b_0 + b_1 x + \cdots + b_t x^t \in \bigcup_n I_n \) and \( b_t \in L_t(I_n) \) for some \( n \), hence \( b_t \in \bigcup_n L_t(I_n) \), and \( L_t(\bigcup_n I_n) \subseteq \bigcup_n L_t(I_n) \). Therefore \( \bigcup_n L_t(I_n) = L_t(\bigcup_n I_n) \).

Proof of theorem 2.5:
Let \( I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots \) be any ascending chain of left ideals of \( R[x] \). Then for all \( t \geq 0 \), \( L_t(I_1) \subseteq L_t(I_2) \subseteq \cdots \subseteq L_t(I_n) \subseteq \cdots \) is an ascending chain of left ideals of \( R \). But \( R \) is a left Quasi-Noetherian ring so there exists \( m \in \mathbb{Z}^+ \) such that \( R^m(\bigcup_n L_t(I_n)) \subseteq L_t(I_m) \) so by Lemma (2.7) \( R^m L_t(\bigcup_n I_n) \subseteq L_t(I_m) \) then by Lemma (2.6) \( (R[x])^m (\bigcup_n I_n) \subseteq I_m \) hence \( R[x] \) is a left Quasi-Noetherian ring.

An immediate consequence of Theorem 2.5 we have the following.
Corollary 2.8:
If $R$ is a left Quasi-Noetherian ring without zero divisors, then $R[x_1, \ldots, x_n]$ is left Quasi-Noetherian.

Next we prove the following

Theorem 2.9:
Let $R$ be a left Quasi-Noetherian ring without zero divisors. Then the ring of formal power series $R[[x]]$ is left Quasi-Noetherian.

Proof:
Imitate the proof of Theorem 2.5 and use $C_t(f) = \{ b_t \in R \mid \exists b_{t+1}, b_{t+2}, \ldots \text{such that } b_t x^f + b_{t+1} x^{f+1} + \ldots \in f \}$ instead of $L_t(f)$.

Let $R$ be a ring then the ring $R\langle x \rangle = \{ \sum_{i \in \mathbb{Z}} a_i x^i \mid a_i \in R \}$ is called the ring of extended formal power series or (Laurent series ring over $R$) [10 P.17]

Now we prove the following

Corollary 2.10:
Let $R$ be a left Quasi-Noetherian ring without zero divisors. Then the ring of extended formal power series $R\langle x \rangle$ is left Quasi-Noetherian.

Proof:
By Theorem 2.5 $R[x]$ is a left Quasi-Noetherian ring. But $R\langle x \rangle$ is a ring of left quotient of $R[x]$ so by Theorem 2.4(a) $R\langle x \rangle$ is a left Quasi-Noetherian ring.

3. The ideal structures

In this section we study the ideal structures of Quasi-Noetherian rings. In particular we prove the following

Theorem 3.1:
Let $R$ be a commutative Quasi-Noetherian ring, then every ideal in $R$ has a primary decomposition.

To prove this we need the following lemmas. But first If $I \triangleleft R$ and $x \in R$, $x \not\in I$ then $I : x = \{ a \in R \mid xa \in I \}$

Lemma 3.2:
Let $R$ be a commutative Quasi-Noetherian ring, $I$ is an ideal in $R$ and $r \in R$, $r \not\in I$. Then the chain $I : r \subseteq I : r^2 \subseteq \cdots \subseteq I : r^n \subseteq \cdots$ terminates and $I = (I : r^t) \cap (I + (r^t))$ for some $t \in \mathbb{Z}^+$. 

Proof:
Since $I : r \subseteq I : r^2 \subseteq \cdots \subseteq I : r^n \subseteq \cdots$ is an ascending chain of ideals in $R$, $R$ is a Quasi-Noetherian ring, then there exists $m \in \mathbb{Z}^+$ such that
Proof of theorem 3.1:

Proof and irreducible ideals. Let $R$ be a commutative Noetherian ring with at least one regular element, hence it is enough to show that the zero ideal of $R$. 

Now it is clear that $I = (I:_R t) \cap (I + (t^r))$, $t = 2m$. On the other hand let $x \in (I:_R t) \cap (I + (t^r))$. Therefore $x = a + t^r b$ for some $a \in I$ and $b \in R$. Hence $t^r x = t^r a + t^{r+1} b \in I$, since $x \in I : t^r$ it follows that $t^r x \in I$. Then $t^{r+1} b \in I$, therefore $b \in I : t^{r+1} = I : t^r$ and $t^r b \in I$. Hence $x \in I$. Thus $I = (I:_R t) \cap (I + (t^r))$.

Lemma 3.3:

Let $Q(R)$ be the ring of quotients of the ring $R$. If $J$ is a primary ideal in $Q(R)$ then $J \cap R$ is a primary ideal in $R$.

Proof:

Let $a, b \in R$, $ab \in J \cap R$. Then $ab \in J$ and $ab \in R$. If $a \notin J \cap R$ so $b \notin J$ but $J$ is a primary ideal in $Q(R)$ therefore $b^n \in J$ hence $b^n \in J \cap R$.

Lemma 3.4:

Let $R$ be a commutative Quasi-Noetherian ring with at least one regular element. Then the zero ideal of $R$ has a primary decomposition.

Proof:

Since $R$ has a ring of quotient $Q(R)$ [3,p62] which is Noetherian and $0_R = 0_{Q(R)}$ therefore $0_{Q(R)} = \bigcap_{i=1}^n J_i$, $J_i$ primary ideal in $Q(R)$, so $0_R = 0_{Q(R)} \cap R = (\bigcap_{i=1}^n J_i) \cap R = \bigcap_{i=1}^n (J_i \cap R)$ and $J_i \cap R$ is primary ideal in $R$ (by lemma 3.3).

Proof of theorem 3.1:

Since the homomorphic image of a Quasi-Noetherian ring is a Quasi-Noetherian, it is enough to show that the zero ideal of $R$ has a primary decomposition. If $0$ is a primary then it is nothing to prove. If $0$ is not primary, let $a, b \in R$ such that $ab = 0$ and $a \neq 0$, $b^n \neq 0$ for any $n \in \mathbb{Z}^+$. Not that $a^n \neq 0$ otherwise $0$ is a primary. Since $R$ is a Quasi-Noetherian ring the chain $0 \subseteq 0: b \subseteq 0: b^2 \subseteq \cdots \subseteq 0: b^n \subseteq \cdots$ terminates and so $0 = (0: b^n) \cap (0 + (b^n)) = \text{ann} b^n \cap (b^n)$ (by lemma 3.2).

Assume that $z = x + b^{n+1}$ where $x \in \text{ann} b^n$ and $b^{n+1} \in \langle b^n \rangle$. If $zy = 0$, $0 \neq y \in R$ then $xy + b^{n+1}y = 0$ but $xy = -b^{n+1}y \in \text{ann} b^n \cap \langle b^n \rangle = 0$, hence $xy = 0$ and $b^{n+1}y = 0$ therefore $y \in \text{ann} b^{n+1} = \text{ann} b^n$, since $x$ is any element in $\text{ann} b^n$, $xy = 0$ for all $y \in \text{ann} b^n$ it follows that $(\text{ann} b^n)^2 = 0$ therefore $\text{ann} b^n$ is a nilpotent ideal which is a contradiction. Therefore $R$ has a regular element, hence $Q(R)$ exists therefore the zero ideal of $R$ has a primary decomposition (by lemma 3.3).

Next we prove the following which gives the relationship between primary and irreducible ideals.
Left quasi-Noetherian modules

Theorem 3.5:
If \( R \) is a commutative Quasi-Noetherian ring, then every irreducible ideal is primary.

Proof:
Let \( I \triangleleft R \) be an irreducible ideal. We need to show that \( I \) is a primary, since the homomorphic image of a Quasi-Noetherian ring is a Quasi-Noetherian, it is enough to show that 0 is a primary ideal. It sufficient to show that if \( xy = 0 \in R \), then either \( x = 0 \) or \( y^n = 0 \) for some \( n \). First we claim that \( \langle x \rangle \cap \langle y^n \rangle = 0 \). To prove this, we consider the ascending chain \( \text{ann}(y) \subseteq \text{ann}(y^2) \subseteq \cdots \subseteq \text{ann}(y^n) \subseteq \cdots \) of ideals of \( R \), since \( R \) is a Quasi-Noetherian ring, then there exists \( m \in \mathbb{Z}^+ \) such that \( R^m(\bigcup_n \text{ann}(y^n)) \subseteq \text{ann}(y^m) = 0: \langle y^m \rangle \) so \( \bigcup_n (\langle y^n \rangle) \subseteq 0: \langle y^m \rangle R^m \subseteq 0: \langle y^{2m} \rangle \) so \( \bigcup_n \text{ann}(y^n) = \text{ann}(y^{2m}) \) and the chain terminates. Therefore \( \text{ann}(y^n) = \text{ann}(y^{n+1}) \). Now if \( a \in \langle x \rangle \cap \langle y^n \rangle \), then \( a = bx = cy^n \) for some \( b, c \) so \( ay = 0 \) since \( b(xy) = b(xy) = 0 \), it follows that \( cy^n y = cy^{n+1} = 0 \) and \( c \in \text{ann}(y^{n+1}) = \text{ann}(y^n) \). Therefore \( a = cy^n = 0 \) and \( \langle x \rangle \cap \langle y^n \rangle = 0 \). Now by irreducibility of the zero ideal, it follows that \( \langle x \rangle = 0 \) or \( \langle y^n \rangle = 0 \). Hence \( x = 0 \) or \( y^n = 0 \).

□

Now we prove the following

Theorem 3.6:
Let \( R \) be a non-nilpotent left Quasi-Noetherian ring. Then there exists a prime ideal in \( R \) which is a right annihilator of some non-zero ideal of \( R \).

Proof:
By Lemma 1.9 \( R \) is not nil, so \( R \) contains at least one non-nilpotent element \( a \) (say), therefore the set \( S = \{a, a^2, \ldots, a^n, \ldots\} \) is closed under multiplication and doesn’t contain zero. Thus there exists a prime ideal \( P \) of \( R \) and \( P \cap S = \emptyset \) by Corollary of [2, p 164]. Now let \( a \in I \triangleleft R \) so \( I \cap r(I_1) = 0 \subseteq P \) and \( I_1 \notin P \)(since \( a \in S \)), it follows that \( r(I_1) \subseteq P \), if \( r(I_1) = P \) then we finish, if not we can choose \( x_1 \in R \) with \( x_1 \notin P \) so if \( x_1 \in I_2 \triangleleft R \) then \( I_1 I_2 \notin P \) but then \( r(I_1 I_2) \subseteq P \). Continuing in this way we have \( r(I_1) \subseteq r(I_1 I_2) \subseteq \cdots \subseteq r(I_1 \cdots I_n) \quad (1) \) which is an ascending chain of right annihilator ideals of \( R \), hence \( 0; I_1 \subseteq 0; I_1 I_2 \subseteq \cdots \subseteq 0; I_1 \cdots I_n \subseteq \cdots \). But \( R \) is a left Quasi-Noetherian ring, hence there exists \( m \in \mathbb{Z}^+ \) such \( R^m(\bigcup_n 0; I_1 \cdots I_n) \subseteq 0; I_1 \cdots I_m \). Therefore \( \bigcup_n (0; I_1 \cdots I_n) \subseteq 0; I_1 \cdots I_m R^m \)
and \( \bigcup_n (0; I_1 \cdots I_n) \subseteq 0; I_1 \cdots I_m I_{m+1} \cdots I_{2m} \) which means that (1) terminates, i.e. \( r(I_1 \cdots I_{2m}) = r(I_1 \cdots I_{2m+1}) = \cdots \). But \( r(I_1 \cdots I_{2m}) \subseteq P \), hence \( l(r(I_1 \cdots I_{2m})) \geq l(P) \), and \( r(I_1 \cdots I_{2m}) = r(l(I_1 \cdots I_{2m})) \subseteq r(l(P)) \). By
maximalty of \( r(l_1 \ldots l_{2m}) \) we have \( P \subseteq r(l(P)) = r(l_1 \ldots l_{2m}) \) Therefore \( r(l_1 \ldots l_{2m}) = P \) and \( P \) is a right annihilator of some non-zero ideal of \( R \).

Finally we prove the following

**Theorem 3.7:**

Let \( R \) be a Quasi-Noetherian ring. Then every ideal of \( R \) contains a product of prime ideals.

**Proof:**

Suppose that the family \( F \) of ideals of \( R \) which doesn't contains any product of prime ideals is non-empty. Therefore if \( I \in F \) then \( I; I \subseteq R \) [Because if \( \prod_{i=1}^{n} P_i \subseteq I, \) For all \( P_i \subseteq I, R \not\supseteq I \) then \( R(\prod_{i=1}^{n} P_i) \subseteq I \). So \( (\prod_{i=1}^{n} P_i)^2 \subseteq I \) for all \( P_i \subseteq I \)]. By Theorem 1.1 \( F \) has a maximal element (say) \( I \). Therefore \( I \) is not a prime ideal, hence there are ideals \( A \) and \( B \) of \( R \) such that \( AB \subseteq I \) but \( A \not\subseteq I \) and \( B \not\subseteq I \). Consider \( (I + A)(I + B) \subseteq I^2 + IB + AI + AB \subseteq I \) since \( I \not\subseteq I + A \), \( I \not\subseteq I + B \) then \( I + A \not\subseteq F, I + B \not\subseteq F \). Thus \( I + A, I + B \) each contain a product of prime ideals and hence so also does \( I \). This is a contradiction. Hence \( F = \emptyset \).

**Corollary 3.8:**

Let \( R \) be a Quasi-Noetherian ring. Then:

(a) There exist a finite number of prime ideals \( P_1, \ldots, P_n \) such that \( \prod_{i=1}^{n} P_i = 0 \)

(b) Every ideal of \( R \) contains power of its radical.

**Proof:**

(a) Follows immediately from Theorem 3.7.

(b) Since \( R \) is Quasi-Noetherian, \( I \not\prec R \), then by Theorem 3.8 \( \prod_{i=1}^{n} P_i \not\subseteq I \) But \( r(I) = \cap_{i=1}^{n} P_i \subseteq P_i \) for all \( i \), then \( (rad(I))^n \subseteq \prod_{i=1}^{n} P_i \subseteq I \).

**References**


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