2-Absorbing Ideals in Semirings

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Abstract

Suppose that $R$ is a commutative semiring with identity $1 \neq 0$. In this paper, we define 2-absorbing ideals in $R$ which is a generalization of prime ideals and prove that subtractive 2-absorbing ideals containing a $Q$-ideal $I$ of $R$ are precisely the subtractive 2-absorbing ideals of the quotient semiring $R/I_Q$ (the difference semiring $R - I$). Also, we prove that if $M$ is monic, 2-absorbing ideal in a polynomial semiring $R[x]$, then each coefficient ideal $M_i$ is a 2-absorbing ideal in $R$ but the converse is not true.

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1. Introduction

A. Badawi [3], introduced the concept of 2-absorbing ideals in a commutative ring with identity element, which is a generalization of prime ideals. In this paper we define 2-absorbing ideals in commutative semirings with identity element and characterize subtractive 2-absorbing ideals containing a $Q$-ideal $I$ of a semiring $R$ and subtractive 2-absorbing ideals of a quotient semiring $R/I_{Q}$ (difference semiring $R - I$). Also, we prove that if $M$ is a monic, 2-absorbing ideal in a polynomial semiring $R[x]$, then each coefficient ideal $M_i$ is a 2-absorbing ideal in $R$ but the converse is not true. For the sake of completeness, we state some definitions and notations used throughout.

Definition 1.1. A non empty set $R$ together with two associative binary operations addition and multiplication is called a semiring if i) addition is a commutative operation ii) there exists $0 \in R$ such that $x + 0 = x = 0 + x$, $x \cdot 0 = 0 = 0 \cdot x$ for each $x \in R$ and iii) multiplication distributes over addition both from left and right.
The concepts of commutative semiring, semiring with identity element, ideal, prime ideal, finitely generated ideal can be defined on the similar lines as in rings.

Throughout this paper we use the following notations

\[ R : \text{commutative semiring with identity element.} \]
\[ \mathbb{Z}_0^+ (\mathbb{N}) : \text{the set of non negative (positive) integers.} \]
\[ (\mathbb{Z}_0^+, +, \cdot) : \text{the semiring of all non negative integers under usual addition and multiplication.} \]
\[ (\mathbb{Z}_0^+ \cup \{\infty\}, \oplus, \odot) : \text{the semiring with identity element } \infty \]
\[ \text{where } a \oplus b = \max\{a, b\} \text{ and } a \odot b = \min\{a, b\}. \]
\[ < a > : \text{the principal ideal generated by } a. \]
\[ < a, b > : \text{the ideal generated by } a \text{ and } b. \]
\[ R[x] : \text{the polynomial semiring over a semiring } R. \]

**Definition 1.2.** An ideal \( I \) of a semiring \( R \) is called subtractive (= \( k \)-ideal) if \( a, a + b \in I, b \in R, \) then \( b \in I. \)

**Theorem 1.3.** ([10], Theorem 2.1) A non-zero ideal \( I \) of \((\mathbb{Z}_0^+, +, \cdot)\) is prime if and only if \( I = < p > \) for some prime number \( p \) or \( I = < 2, 3 >. \)

**Definition 1.4.** A proper ideal \( I \) of a semiring \( R \) is called 2-absorbing ideal if whenever \( a, b, c \in R \) and \( abc \in I, \) then \( ab \in I \) or \( ac \in I \) or \( bc \in I. \)

Clearly, every prime ideal of a semiring \( R \) is a 2-absorbing ideal.

**Example 1.5.** In the semiring \((\mathbb{Z}_0^+, +, \cdot), \) let \( I = < 4, 5 > = \{0, 4, 5, 8, 9, 10, 12, 13, 14, \ldots\} = \mathbb{Z}_0^+ - \{1, 2, 3, 6, 7, 11\}. \) Then \( I \) is a 2-absorbing ideal but by Theorem 1.3, \( I \) is not a prime ideal.

**Theorem 1.6.** Every ideal of \( R = (\mathbb{Z}_0^+ \cup \{\infty\}, \oplus, \odot) \) is 2-absorbing.

*Proof.* \( I \) is an ideal of \( R \) if and only if \( I = \{0, 1, 2, \ldots, t\} = I_t \) for some \( t \in \mathbb{Z}_0^+ \) or \( I = \mathbb{Z}_0^+ \) or \( I = R \) ([6], Theorem 5). If \( a \odot b \odot c \in I, \) then \( a \) or \( b \) or \( c = \min\{a, b, c\} = a \odot b \odot c \in I. \) Hence \( a \odot b \) or \( a \odot c \) or \( b \odot c \in I. \)

If \( I \) is an ideal of a semiring \( R, \) then the radical of \( I \) is

\[ \sqrt{I} = \{a \in R : a^n \in R \text{ for some } n \in \mathbb{N}\} \]

**Theorem 1.7.** If \( I \) is a 2-absorbing ideal of a semiring \( R, \) then \( \sqrt{I} \) is a 2-absorbing ideal of \( R. \)

*Proof.* Let \( I \) be a 2-absorbing ideal of \( R. \) If \( x \in \sqrt{I}, \) then \( x^n \in I \) for some \( n \in \mathbb{N}. \) Since \( I \) is a 2-absorbing ideal, \( x^2 \in I. \) Let \( a, b, c \in R \) and \( abc \in \sqrt{I}. \) Then \( a^2b^2c^2 = (abc)^2 \in I. \) Since \( I \) is a 2-absorbing ideal, we may assume that \( (ab)^2 = a^2b^2 \in I. \) Now \( ab \in \sqrt{I}. \) Hence \( \sqrt{I} \) is a 2-absorbing ideal.
Converse of the Theorem 1.7 is not true.

**Example 1.8.** In the semiring \((\mathbb{Z}_0^+, +, \cdot)\), let \(I = < 3, 5 > = \{0, 3, 5, 6, 8, 9, 10, \ldots\} = \mathbb{Z}_0^+ - \{1, 2, 4, 7\}\). Then \(I\) is not a 2-absorbing ideal as \(2 \cdot 2 \cdot 2 = 8 \in I\) but \(2 \cdot 2 \notin I\). Clearly \(\sqrt{I} = \mathbb{Z}_0^+ - \{1\} = < 2, 3 >\). By Theorem 1.3, \(I\) is a prime ideal in \(\mathbb{Z}_0^+\) and hence it is a 2-absorbing ideal.

2. 2-ABSORBING IDEALS IN QUOTIENT SEMIRINGS

**Definition 2.1.** ([1]) An ideal \(I\) of a semiring \(R\) is called a \(Q\)-ideal if there exists a subset \(Q\) of \(R\) such that

1) \(R = \cup \{q + I : q \in Q\}\).
2) if \(q_1, q_2 \in Q\), then \((q_1 + I) \cap (q_2 + I) \neq \emptyset \iff q_1 = q_2\).

Let \(I\) be a \(Q\)-ideal of a semiring \(R\). Then \(R/I(Q) = \{q + I : q \in Q\}\) forms a semiring under the following addition \(\oplus\) and multiplication \(\odot\), \((q_1 + I) \oplus (q_2 + I) = q_3 + I\) where \(q_3 \in Q\) is unique such that \(q_1 + q_2 + I \subseteq q_3 + I\) and \((q_1 + I) \odot (q_2 + I) = q_4 + I\) where \(q_4 \in Q\) is unique such that \(q_1q_2 + I \subseteq q_4 + I\). This semiring \(R/I(Q)\) is called a quotient semiring of \(R\) by \(I\) and denoted by \((R/I(Q), \oplus, \odot)\) or just \(R/I(Q)\).

**Lemma 2.2.** Let \(I\) be a \(Q\)-ideal of a semiring \(R\) and \(a, b \in R\). Let \(q, q', q'', q''' \in Q\) be unique elements such that \(a + I \subseteq q + I\), \(b + I \subseteq q' + I\), \(q + q' + I \subseteq q'' + I\), \(qq' + I \subseteq q''' + I\). Then \(a + b \in q'' + I\) and \(ab \in q''' + I\).

**Proof.** Trivial. \(\square\)

**Lemma 2.3.** ([2], Lemma 2.1) Let \(R\) be a semiring, \(I\) a \(Q\)-ideal of \(R\) and \(A\) a subtractive ideal of \(R\) with \(I \subseteq A\). Then \(I = Q \cap A\)-ideal of \(A\).

**Theorem 2.4.** ([2], Proposition 2.2 and Theorem 2.3) Let \(R\) be a semiring, \(I\) a \(Q\)-ideal of \(R\). Then \(A\) is a subtractive ideal of \(R\) with \(I \subseteq A\) if and only if \(A/I(Q \cap A)\) is a subtractive ideal in \(R/I(Q)\).

**Theorem 2.5.** Let \(I\) be a \(Q\)-ideal in a semiring \(R\) and \(A\) be an ideal of \(R\) such that \(I \subseteq A\). Then \(A\) is a subtractive 2-absorbing ideal if and only if \(A/I(Q \cap A)\) is a subtractive 2-absorbing ideal in \(R/I(Q)\).

**Proof.** Let \(A\) be a subtractive 2-absorbing ideal of \(R\) and \(q_1 + I, q_2 + I, q_3 + I \in R/I(Q)\) be such that \((q_1 + I) \odot (q_2 + I) = q + I \in A/I(Q \cap A)\) where \(q \in Q \cap A\) is a unique element such that \(q_1q_2q_3 + I \subseteq q + I\). Therefore \(q_1q_2q_3 = q + i \in A\) for some \(i \in I \subseteq A\). Since \(A\) is a 2-absorbing ideal, we may assume that \(q_1q_2 \in A\). Let \((q_1 + I) \odot (q_2 + I) = q_4 + I\) where \(q_4 \in Q\) is a unique element such that \(q_1q_2 + I \subseteq q_4 + I\). Hence \(q_1q_2 = q_4 + i_1\) for some \(i_1 \in I \subseteq A\). Since \(A\) is subtractive and \(q_1q_2 \in A\), \(q_4 \in A\) and hence \(q_4 \in Q \cap A\). So \((q_1 + I) \odot (q_2 + I) = q_4 + I \in A/I(Q \cap A)\). Hence \(A/I(Q \cap A)\) is a 2-absorbing ideal in \(R/I(Q)\). Also by Theorem 2.5, \(A/I(Q \cap A)\) is a subtractive ideal in \(R/I(Q)\). Conversely, assume that \(A/I(Q \cap A)\) is a subtractive 2-absorbing ideal in \(R/I(Q)\). By Theorem 2.5, \(A\) is a subtractive ideal. Let \(a_1a_2a_3 \in A\)
Proof. By Lemma 2.3 and Theorem 2.7 we have, 

I

is a subtractive ideal of

q

M

R

subtractive ideal in

by Lemma 2.2, there exists

i

following theorem.

prime ideal of

R

ring

ideal, we may assume that

a

2-absorbing ideal.

Since

A

R

we shall also write this as

R

semiring under addition and multiplication. it is called the difference semiring

−

A

L. Dale (\cite{5}, Theorem 3.4) has also proved that

there exists

f

A

L. Dale \cite{5} has proved that if

A

is a monic ideal

in

R

2-absorbing ideals in polynomial semirings

are called coefficient ideals. L. Dale (\cite{5}, Theorem 3.4) has also proved that

there exists

x

y

I

A

\mathbb{R}

−
o

prime ideal in

R

but the converse is not true. In this section we prove the following theorem.

Theorem 2.7. (\cite{8}, Theorem 2.4) Let

I

be a Q-ideal in a semiring

R

Then the difference semiring

R − I

and the quotient semiring

R/I_Q

are isomorphic.

Corollary 2.8. Let

I

be a Q-ideal in a semiring

R

and

A

a subtractive ideal of

I ⊆ A

. Then

A

is a subtractive 2-absorbing ideal in

R

if and only if

A − I

is a subtractive 2-absorbing ideal in

R − I

.

Proof. By Lemma 2.3 and Theorem 2.7 we have,

I

is a Q-ideal in

R

and

A

is a subtractive ideal of

R

with

I ⊆ A

implies

Q ∩ A

ideal in

A

and

A − I

≅ A/I_{Q ∩ A}

. Thus by Theorem 2.5, \( A \) is a subtractive 2-absorbing ideal in

R ⇔ A/I_{Q ∩ A}

is a subtractive 2-absorbing ideal in

R/I_Q

⇒ A − I

is a subtractive 2-absorbing ideal in

R − I

.

3. 2-ABSORBING IDEALS IN POLYNOMIAL SEMIRINGS

L. Dale \cite{5} has proved that if

A

is an ideal in

R[x]

and

A_i

= \{a ∈ R : there exists

f(x) ∈ A

such that

ax^i

is a term of

f(x)\}, then

\{A_n\}

is an ascending chain of ideals in

R

(\cite{5}, Theorem 2.2). The ideals

\{A_n\}

are called coefficient ideals. L. Dale (\cite{5}, Theorem 3.4) has also proved that

a monic ideal

M

in

R[x]

is a subtractive ideal if and only if each

M_i

is a subtractive ideal in

R

. J. N. Chaudhari and V. Gupta (\cite{4}, Theorem 3.1) have proved that if a monic ideal

M

is a prime ideal in

R[x]

, then each

M_i

is a prime ideal of

R

, but the converse is not true. In this section we prove the following theorem.

Theorem 3.1. Let

M

be a monic, 2-absorbing ideal in a polynomial semiring

R[x]

. Then each coefficient ideal

M_i

is a 2-absorbing ideal in

R

.
Proof. Let \( M \) be a monic, 2-absorbing ideal of \( R[x] \) and \( abc \in M_i \). Then there exists \( f(x) \in M \) such that \( (abc)x^i \) is a term of \( f(x) \). Since \( M \) is monic \( (abc)x^i = a \cdot b \cdot cx^i \in M \). Since \( M \) is a 2-absorbing ideal, \( ab \in M \) or \( acx^i \in M \) or \( bcx^i \in M \). Hence \( ab \in M_0 \subseteq M_i \) or \( ac \in M_i \) or \( bc \in M_i \). Thus each coefficient ideal \( M_i \) is a 2-absorbing ideal. \( \square \)

Converse of the Theorem 3.1 is not true.

**Example 3.2.** Let \( R = (\mathbb{Z}_0^+ \cup \{\infty\}, \oplus, \odot) \) be a commutative semiring with identity \( \infty \). Define \( M = \{ f(x)x^3 \in R[x] : f(x) = \sum_{i=0}^{n} a_i x^i \text{ where } a_i \leq i + 1 \text{ for all } i \} \). Then we will show that \( M \) is a monic ideal and each coefficient ideal \( M_i \) is a 2-absorbing ideal but \( M \) is not a 2-absorbing ideal.

1) \( M \) is an ideal : Let \( f(x) = \sum_{i=0}^{n} a_i x^i x^3, g(x) = \sum_{i=0}^{n} b_i x^i x^3 \in M \) where \( a_i, b_i \leq i + 1 \). Therefore \( f(x) + g(x) = \sum_{i=0}^{n} (a_i \oplus b_i)x^i x^3 \in M \), since \( a_i \oplus b_i = \max\{a_i, b_i\} \leq i + 1 \). If \( h(x) = \sum_{j=0}^{m} c_j x^j \in R[x], \) then \( f(x)h(x) = (\sum_{i=0}^{n} a_i x^i x^3)(\sum_{j=0}^{m} c_j x^j) = \sum_{k=0}^{n+m} d_k x^k x^3 \) where \( d_k = \oplus \sum_{k=i+j} a_i \odot c_j \) where \( a_i \odot c_j = \min\{a_i, c_j\} \leq a_i \leq i + 1 \leq k + 1 \). Hence \( d_k \leq k + 1 \) for all \( k \). Now \( f(x)h(x) \in M \).

2) \( M \) is a monic ideal : Let \( f(x) = \sum_{i=0}^{n} a_i x^i x^3 \in M \). Then each term \( a_i x^i x^3 = a_i x^i x^3 \in M, \) since \( a_i \leq i + 1 \) for all \( i \).

3) Each \( M_t \) is a 2-absorbing ideal : Clearly \( M_0 = M_1 = M_2 = \{0\} \). For \( t \geq 3 \),

\[
M_t = \{ a \in R : \text{there exists } g(x) = \sum_{i=0}^{n} a_i x^i x^3 \in M \text{ such that } ax^t \text{ is a term of } g(x) \} \\
= \{ a \in R : \text{there exists } g(x) = \sum_{i=0}^{n} a_i x^{i+3} \in M \text{ such that } ax^t = a_{t-3}x^t \text{ is a term of } g(x) \text{ where } a = a_{t-3} \leq t - 3 + 1 = t - 2 \text{ for all } t \} \\
= \{ a \in R : a \leq t - 2 \} \\
= \{ 0, 1, 2, \ldots, t - 2 \}
\]
By Theorem 1.6, each $M_t$ is a 2-absorbing ideal of $R$.

4) $M$ is not a 2-absorbing ideal : $x \cdot x \cdot x = 1 \cdot x^3 \in M$ where $a_0 = 1 \leq 0 + 1$ and $a_0 \in R[x]$ but $x \cdot x = 1 \cdot x^2 \notin M$.

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