A Note on Strongly Euclidean Semirings

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Abstract. Theory of ideals in the semiring $\mathbb{Z}_0^+$ was given by P. J. Allen and L. Dale [2] and they proved that $\mathbb{Z}_0^+$ is a Noetherian semiring. Further, characterization of subtractive ideals and prime ideals in the semiring $\mathbb{Z}_0^+$ has been given by V. Gupta and J. N. Chaudhari ([3], [7]). In this paper, we study ideal theory in the semiring $(\mathbb{Z}_0^+, \text{gcd, lcm})$ and obtain characterizations of $Q$-ideals, prime ideals, maximal ideals and primary ideals. Also it is proved that, if $R$ is a strongly Euclidean IS-semiring, then $R$ and $R_{n \times n}$ are principal ideal semirings.

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1. Introduction

A non-empty set $R$ together with two associative binary operations addition and multiplication is called a semiring if i) addition is a commutative operation ii) there exists $0 \in R$ such that $x+0 = x = 0+x$, $x \cdot 0 = 0 = 0 \cdot x$ for each $x \in R$ and iii) multiplication distributes over addition both from left and right. The concept of ideal, finitely generated ideal, principal ideal, prime ideal, maximal ideal, semiprime ideal, primary ideal in a commutative semiring with identity 1 can be defined on the similar lines as in commutative rings with identity 1. All semirings are assumed to be semirings with identity element. $\mathbb{Z}_0^+$ ($\mathbb{N}$) will denote the set of all non-negative (positive) integers. An ideal $I$ of a semiring $R$ is called (1) subtractive ideal (= k-ideal) if $a, a + b \in I, b \in R, \text{ then } b \in I$. (2) $Q$-ideal (= partitioning ideal) if there exists a subset $Q$ of $R$ such that

1. $R = \cup\{q + I : q \in Q\}$.
2. if $q_1, q_2 \in Q, \text{ then } (q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Lemma 1.1. ([1], Lemma 7) Let $I$ be a $Q$-ideal of a semiring $R$. If $x \in R$, then there exists a unique $q \in Q$ such that $x + I \subseteq q + I$. 
Theorem 1.2. ([6], Theorem 1.4) An ideal I of a strongly Euclidean semiring R is Q-ideal if and only if I is principal ideal.

Lemma 1.3. ([3], Page 648) A is an ideal of the matrix semiring $R_{n \times n}$ if and only if there exists an ideal I of R such that $A = I_{n \times n}$.

2. Ideals in the Semiring $(\mathbb{Z}_0^+, \oplus, \odot)$

For $a, b \in \mathbb{Z}_0^+$, we define,

1) $a \oplus b = \gcd \{a, b\}$ if $a, b \in \mathbb{N}$;
2) $a \odot b = \text{lcm} \{a, b\}$ if $a, b \in \mathbb{N}$;
3) $a \oplus 0 = a$ and $a \odot 0 = 0$ for all $a \in \mathbb{Z}_0^+$;
4) $ab = \text{usual product of } a \text{ and } b$;
5) $a^n = \text{aa...aa}(n\text{-times})$.

Clearly $(\mathbb{Z}_0^+, \oplus, \odot)$ is a commutative semiring with identity element 1. For $a \in (\mathbb{Z}_0^+, \oplus, \odot)$, we denote, $\langle a \rangle = \{n \odot a : n \in \mathbb{Z}_0^+\}$, the principal ideal generated by a.

Lemma 2.1. If $a \in (\mathbb{Z}_0^+, \oplus, \odot)$, then $\langle a \rangle = \{na : n \in \mathbb{Z}_0^+\}$.

Proof. We have $na = na \odot a \in \langle a \rangle$. Thus $\{na : n \in \mathbb{Z}_0^+\} \subseteq \langle a \rangle$. On the other hand, if $x \in \langle a \rangle$, then there exists $n \in \mathbb{Z}_0^+$ such that $x = n \odot a = ka$ for some $k \in \mathbb{Z}_0^+$. So $\langle a \rangle \subseteq \{na : n \in \mathbb{Z}_0^+\}$. □

Lemma 2.2. Every ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$ is a principal ideal.

Proof. Let $I$ be a non-zero ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$ and choose least non-zero element $d \in I$. Claim: $I = \langle d \rangle$. If $a \in I$, then $a \oplus d \in I$ and $a \odot d = d$ as $d$ is the least non-zero element of $I$. Now $a = kd$ for some $k \in \mathbb{Z}_0^+$. By Lemma 2.1, $a \in \langle d \rangle$. Hence $I \subseteq \langle d \rangle$. On the other hand, for any $n \in \mathbb{Z}_0^+$, $nd = n \odot d \in I$. By Lemma 2.1, $\langle d \rangle \subseteq I$. □

Theorem 2.3. $(\mathbb{Z}_0^+, \oplus, \odot)$ is a Noetherian semiring.

Lemma 2.4. Every ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$ is subtractive.

Proof. Let $I$ be an ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$. By Lemma 2.2, $I = \langle d \rangle$ for some $d \in I$. If $a, b \in I = \langle d \rangle$, then $a \oplus b = \gcd\{a, b\}$ are multiples of $d$ and hence $b$ is a multiple of $d$. By Lemma 2.1, $b \in \langle d \rangle = I$. □

Lemma 2.5. If $I$ is a non-zero proper ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$, then $I$ is not a Q-ideal.

Proof. Let $I$ be a non-zero proper ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$. By Lemma 2.2, $I = \langle d \rangle$ for some $d \in \mathbb{Z}_0^+ - \{0, 1\}$. Take $d = p_1^{r_1}p_2^{r_2}...p_k^{r_k}$ where $p_1, p_2,..., p_k$ are pairwise distinct primes and $r_i \in \mathbb{N}$. Suppose that $I$ is a Q-ideal. We claim that there exists a unique $q \in Q$ such that $p \in q \oplus I$ for all primes $p$ other than $p_i$. Let $p', p''$ be any distinct primes other than $p_i$. By Lemma 1.1, there are unique $q_1, q_2 \in Q$ such that $p' \in p' \oplus I \subseteq q_1 \oplus I$ and $p'' \in p'' \oplus I \subseteq q_2 \oplus I$. Since $p', p''$ are primes other than $p_i$, $1 = p' \oplus d \in q_1 \oplus I$ and $1 = p'' \oplus d \in q_2 \oplus I$.
Hence $(q_1 \oplus I) \cap (q_2 \oplus I) \neq \emptyset$. Since $I$ is a $Q$-ideal, $q_1 = q_2 = q$ say. Now we have a unique $q \in Q$ such that $p \in q \oplus I$ for all primes $p$ other than $p_i$'s. Clearly $q \geq 1$. By above claim choose a prime $f > q$ such that $f \in q \oplus I$. By Lemma 2.1, $f = q \oplus nd$ for some $n \in \mathbb{Z}_0^+$. So $f \mid q$, a contradiction. Therefore $I$ is not a $Q$-ideal.

**Theorem 2.6.** \{0\} and $\mathbb{Z}_0^+$ are the only $Q$-ideals in the semiring $(\mathbb{Z}_0^+, \oplus, \odot)$.

**Proof.** Let $I$ be an ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$. If $I = \{0\}$, then clearly $I$ is a $Q$-ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$ with $Q = \mathbb{Z}_0^+$. If $I = \mathbb{Z}_0^+$, then $I$ is a $Q$-ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$ with $Q = \{0\}$. 

**Theorem 2.7.** $I$ is a non-zero prime ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$ if and only if $I = \langle p^r \rangle$ for some prime $p$ and $r \geq 1$.

**Proof.** Let $I$ be a non-zero prime ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$. By Lemma 2.2, $I = \langle d \rangle$ where $d = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where $p_1, p_2, \ldots, p_k$ are pairwise distinct primes and $r_i \in \mathbb{N}$. If $k \geq 2$, then $p_1^{r_1} \odot (p_2^{r_2} \cdots p_k^{r_k}) = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is in $I$ but $p_1^{r_1} \notin I$ and $p_2^{r_2} \cdots p_k^{r_k} \notin I$, a contradiction to $I$ is a prime ideal. Hence $k = 1$. Now $d = p_1^{r_1}$. Conversely, let $I = \langle p^r \rangle$ for some prime $p$ and $r \geq 1$ and let $a \odot b \in I = \langle p^r \rangle$. By Lemma 2.1, $p^r \mid \text{lcm}\{a, b\}$ implies $p^r \mid a$ or $p^r \mid b$. Again by Lemma 2.1, $a \in I$ or $b \in I$ and hence $I$ is a prime ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$.

**Theorem 2.8.** $I$ is a non-zero maximal ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$ if and only if $I = \langle p^r \rangle$ for some prime $p$.

**Proof.** Let $I$ be a non-zero maximal ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$. By Lemma 2.2, $I = \langle d \rangle$ for some $d \in \mathbb{Z}_0^+$. If $d$ is not prime, then $d = pq$ for some $1 < p < d$ and $1 < q < d$. But then $I = \langle d \rangle \subseteq \langle p \rangle \subseteq \mathbb{Z}_0^+$, a contradiction to $I$ is a maximal ideal. Hence $d$ is a prime number. Conversely, suppose that $I = \langle p^r \rangle$ for some prime $p$. Let $J$ be any ideal of $\mathbb{Z}_0^+$ such that $I \subseteq J \subseteq \mathbb{Z}_0^+$. By Lemma 2.2, $J = \langle d \rangle$ for some $d > 1$. Since $\langle p^r \rangle = I \subseteq J = \langle d \rangle$, $d = p$. Hence $I$ is a maximal ideal.

**Theorem 2.9.** Every ideal of the semiring $(\mathbb{Z}_0^+, \oplus, \odot)$ is semiprime.

**Proof.** Let $I$ be a non-zero ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$ and $a \odot a \in I$. But then $a \in I$.

**Theorem 2.10.** A non-zero ideal $I$ of the semiring $(\mathbb{Z}_0^+, \oplus, \odot)$ is primary if and only if it is a prime ideal.

**Proof.** Let $I$ be a primary ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$ and $a \odot b \in I$. Therefore $a \in I$ or $b \odot b \odot b \odot \ldots \odot b \in I$ i.e. $a \in I$ or $b \in I$. Hence $I$ is a prime ideal. Converse is trivial.

### 3. Strongly Euclidean Semirings

**Definition 3.1.** A semiring $R$ is called IS-semiring if every ideal of $R$ is subtractive.
Example 3.2. By Lemma 2.4, the semiring \((\mathbb{Z}_0^+, \oplus, \odot)\) is a \(\text{IS-semiring}\). By Proposition ([3], Proposition 2.19), the semiring \((\mathbb{Z}_0^+, +, \cdot)\) is not \(\text{IS-semiring}\).

Definition 3.3. A semiring \(R\) is called principal ideal semiring (\(\text{PIS}\)) if every ideal of \(R\) is principal ideal.

Definition 3.4. A commutative semiring \(R\) is called strongly Euclidean if there exists a function \(d : R - \{0\} \rightarrow \mathbb{Z}_0^+\) such that (1) \(d(ab) \geq d(a)\) for all \(a, b \in R - \{0\}\) and (2) if \(a, b \in R\) with \(b \neq 0\), then there exist unique \(q, r \in R\) such that \(a = bq + r\) where either \(r = 0\) or \(d(r) < d(b)\).

Theorem 3.5. Every strongly Euclidean \(\text{IS-semiring}\) is a principal ideal semiring.

\textbf{Proof.} Let \(R\) be a strongly Euclidean \(\text{IS-semiring}\) with function \(d\) and \(I\) an ideal of \(R\), \(I \neq 0\). Let \(A = \{d(a) \in \mathbb{Z}_0^+ : a \in I - \{0\}\}\). Then \(A\) has the least element say \(d(a)\). We claim that \(I = \langle a \rangle\). Let \(x \in I\). Then there exist unique \(q, r \in R\) such that \(x = aq + r\) where \(r = 0\) or \(d(r) < d(a)\). If \(r \neq 0\), then \(r \in I\), since \(I\) is a subtractive ideal. As \(d(a)\) is the least element of \(A\), \(d(a) \leq d(r)\), a contradiction. Hence \(r = 0\). Now \(x = aq \in \langle a \rangle\). Thus \(I \subseteq \langle a \rangle\). But \(\langle a \rangle \subseteq I\). So \(I = \langle a \rangle\). Hence \(R\) is a principal ideal semiring.

Converse of the Theorem 3.5 is not true.

Example 3.6. By Lemma 2.2, \(R = (\mathbb{Z}_0^+, \oplus, \odot)\) is a \(\text{PIS}\). If \(R\) is strongly Euclidean semiring, then by Theorem 1.2, every principal ideal of \(R\) is a \(Q\)-ideal, a contradiction to Lemma 2.5. Hence \(R\) is not strongly Euclidean semiring.

Example 3.7. The semiring \((\mathbb{Z}_0^+, +, \cdot)\) is a strongly Euclidean semiring but not \(\text{PIS}\).

Example 3.8. The semiring \(R = (\mathbb{Z}_0^+ \cup \{\infty\}, \max, \min)\) is \(\text{IS-semiring}\). By Theorem ([5], Theorem 5), \(I = \mathbb{Z}_0^+\) is not a principal ideal and hence \(R\) is not a \(\text{PIS}\). So by Theorem 3.5, \(R\) is not a strongly Euclidean semiring.

Theorem 3.9. If \(R\) is a strongly Euclidean \(\text{IS-semiring}\), then \(R_{n \times n}\) is a \(\text{PIS}\).

\textbf{Proof.} Let \(R\) be a strongly Euclidean \(\text{IS-semiring}\) and \(A\) be any ideal of \(R_{n \times n}\). By Lemma 1.3, \(A = I_{n \times n}\) for some ideal \(I\) of \(R\). By Theorem 3.5, \(R\) is \(\text{PIS}\). So \(I\) is a principal ideal say \(I = \langle a \rangle\). We claim that \(A = \langle B \rangle\) where \(B = \begin{bmatrix} a & 0 & \ldots & 0 \\ 0 & a & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{bmatrix}\). Let \(X = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{bmatrix} \in A = I_{n \times n}\). Therefore \(x_{ij} \in I = \langle a \rangle\). So \(x_{ij} = t_{ij}a\) where \(t_{ij} \in R\) for all \(i, j\). Take \(T = \begin{bmatrix} t_{11} & t_{12} & \ldots & t_{1n} \\ t_{21} & t_{22} & \ldots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \ldots & t_{nn} \end{bmatrix}\)
Then $X = TB \in <B>$. Thus $A \subseteq <B>$. Other inclusion is trivial. Hence $A = <B>$. Thus $R_{n \times n}$ is a PIS.

References


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