\textbf{FI-Injective Resolutions and Dimensions}

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Abstract

In this paper, we show the existence of FI-injective preenvelopes over coherent rings, characterize FI-injective resolutions and define FI-injective dimension for modules and rings. It measures how far away a left module from being FI-injective module.

Mathematics Subject Classification: 16E10, 16D50

Keywords: Coherent ring, Preenvelope, FP-injective module, FI-injective module, FI-injective dimension

1 Introduction

We first recall some known notions and facts which we need in the sequel.

Let $R$ be a ring. A left $R$-module $M$ is called FP-injective \[9\] if $\text{Ext}^1_R(N, M) = 0$ for all finitely presented left $R$-modules $N$. A left $R$-module $M$ is called FI-injective $[8]$ if $\text{Ext}^1_R(N, M) = 0$ for any FP-injective left $R$-module $N$. Let $\mathcal{C}$ be a class of right $R$-modules. Following $[4]$, a homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a \(\mathcal{C}\)-preenvelope of $M$, if for any homomorphism $f : M \to F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such $g$ are automorphisms of $F$ when $F' = F$ and $f = \phi$, the $\mathcal{C}$-preenvelope $\phi$ is called a $\mathcal{C}$-envelope of $M$. Dually, we have the definitions of a $\mathcal{C}$-precovers and a $\mathcal{C}$-covers.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of $R$-modules is called a cotorsion theory if $\mathcal{F}^\perp = \mathcal{C}$ and $\mathcal{C}^\perp = \mathcal{F}$, where $\mathcal{F}^\perp = \{ F : \text{Ext}^1_R(F, C) = 0 \text{ for all } F \in \mathcal{F} \}$, and $\mathcal{C}^\perp = \{ F : \text{Ext}^1_R(F, C) = 0 \text{ for all } C \in \mathcal{C} \}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called perfect if every $R$-module has a $\mathcal{C}$-envelope and an $\mathcal{F}$-cover.

In what follows, let $\mathcal{FI}(\mathcal{FP})$ be the class of all FI-injective (FP-injective) left $R$-modules.
In section 2, we show that if $R$ is a left coherent and left self-$FP$-injective ring, then $(\mathcal{FP}, \mathcal{FI})$ is a perfect cotorsion theory. In particular, every left $R$-module has a special $\mathcal{FI}$-preenvelope, and every left $R$-module has a special $\mathcal{FP}$-precover.

In section 3, the definition and some general results are given. For a left $R$-module $M$, we define the $\mathcal{FI}$-injective dimension $\text{fid}(M)$ of $M$ to the smallest integer $n \geq 0$ such that $\text{Ext}_{R}^{n+1}(M, N) = 0$ for any $FP$-injective left $R$-module $N$. The left $FI$-injective dimension $\text{lfid}(R)$ of a ring $R$ is defined as $\sup\{\text{fid}(M) : M \text{ is any left } R\text{-module}\}$. It is shown that for a left coherent ring $R$, $\text{lfid}(R) = \sup\{\text{fid}(M) : M \text{ is any left } R\text{-module}\}$.

Throughout this article, $R$ is an associative ring with identity and all modules are unitary. For an $R$-module $M$, the character module $\text{Hom}_{Z}(M, Q/Z)$ is denoted by $M^{+}$, $fd(M)$, $id(M)$ and $FP - id(M)$ denote the flat, injective and $FP$-injective dimension of $M$, respectively. General background materials can be found in [2] and [4].

2 $FI$-injective Envelopes

Lemma 2.1 ([5]) Let $R$ be a left coherent ring and $M$ a left $R$-module. Then $fd(M^{+}) = FP - id(M)$.

Lemma 2.2 ([1]) Let $\mathcal{F}$ be a class of modules closed under direct sums, extensions, continuous well-ordered unions, and contain all projective modules. If $\mathcal{F}^{\perp} = S^{\perp}$ for a set $S \subseteq \mathcal{F}$, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is a cotorsion theory.

Recall that a ring $R$ is called left self-$FP$-injective [9] if $_RR$ is an $FP$-injective module. In what follows, let $\mathcal{FI}$ ($\mathcal{FP}$) be the class of all $FI$-injective ($FP$-injective) left $R$-modules. Now we have the following theorem

Theorem 2.3 If $R$ is a left coherent and left self-$FP$-injective ring, then $(\mathcal{FP}, \mathcal{FI})$ is a perfect cotorsion theory. In particular, every left $R$-module has a special $\mathcal{FI}$-preenvelope, and every left $R$-module has a special $\mathcal{FP}$-precover.

Proof: Let $\text{Card}(R) = \aleph_{\beta}$ and $F \in \mathcal{FP}$. By [4, Lemma 5.3.12], for each $x \in F$, there is a pure submodule $S$ of $F$ with $x \in S$ such that $\text{Card}(S) \leq \aleph_{\beta}$ (simply let $N = Rx$ and $f = id_{N}$ in the Lemma). So we can write $F$ as a union of a continuous chain $(F_{\alpha})_{\alpha < \lambda}$ of pure submodules of $F$ such that $\text{Card}(F_{0}) \leq \aleph_{\beta}$ and $\text{Card}(F_{\alpha+1}/F_{\alpha}) \leq \aleph_{\beta}$ whenever $\alpha + 1 < \lambda$. If $N$ is a left $R$-module such that $\text{Ext}_{R}^{1}(F_{0}, N) = 0$ and $\text{Ext}_{R}^{1}(F_{\alpha+1}/F_{\alpha}, N) = 0$ whenever $\alpha + 1 < \lambda$, then $\text{Ext}_{R}^{1}(F, N) = 0$ by [4, Theorem 7.3.4]. Since $F_{\alpha}$ is a pure
submodule of $F$ for any $\alpha < \lambda$, $F^+ \rightarrow F^+_\alpha \rightarrow 0$ is split. Then $F^+_\alpha \in \mathcal{F}$ since $F^+ \in \mathcal{F}$ by Lemma 2.1, and so $F_\alpha \in \mathcal{FP}$ by Lemma 2.1 again. On the other hand, $F_\alpha$ is a pure submodule of $F_{\alpha+1}$ whenever $\alpha + 1 < \lambda$, so the exact sequence $0 \rightarrow F_\alpha \rightarrow F_{\alpha+1} \rightarrow F_{\alpha+1}/F_\alpha \rightarrow 0$ induces the split exact sequence $0 \rightarrow (F_{\alpha+1}/F_\alpha)^+ \rightarrow F^+_{\alpha+1} \rightarrow F^+_\alpha \rightarrow 0$. Thus $(F_{\alpha+1}/F_\alpha)^+ \in \mathcal{F}$ since $F^+_{\alpha+1} \in \mathcal{F}$ by Lemma 2.1, and hence $F_{\alpha+1}/F_\alpha \in \mathcal{FP}$. Let $X$ be a set of representatives of all modules $G \in \mathcal{FP}$ with $\text{Card}(G) \leq \aleph_\beta$. Then $\mathcal{FI} = X^{\perp}$.

We note that $\mathcal{FP}$ is closed under direct sums, extensions, direct limits since $R$ is left coherent, and contains all projective modules since $R$ is self-$\mathcal{FP}$-injective ring. Therefore $(\mathcal{FP}, \mathcal{FI})$ is a cotorsion theory by Lemma 2.2.

Since $(\mathcal{FP}, \mathcal{FI})$ is cogenerated by the set $X$, $(\mathcal{FP}, \mathcal{FI})$ is a complete cotorsion theory by [3, Theorem 10]. Moreover, $(\mathcal{FP}, \mathcal{FI})$ is a perfect cotorsion theory by [4, Theorem 7.2.6] (for $\mathcal{FP}$ is closed under direct limits).

Remark 2.4 (1) We note that Theorem 2.3 extends the work of [1, Theorem 2.8], where the same result is obtained under the hypothesis that $R$ is left Noetherian.

(2) Choose a field $F$, and set $F_i = F$ for $i = 1, 2, \ldots$, $S = \prod_{i=1}^\infty F_i$. Then $S$ is a commutative Von Neumann regular ring. Let $R = S$, the ring of polynomials indeterminates over $S$, then $R$ is a coherent ring with $wD(R) = 0$ (see [6]). Clearly, the ring $R$ satisfies the condition of Theorem 2.3, but it is not Noetherian.

3 FI-injective Dimensions

Definition 3.1 Let $R$ be a ring. For a left $R$-module $M$, let $\text{fid}(M)$ denote the smallest integer $n \geq 0$ such that $\text{Ext}^{n+1}_R(N, M) = 0$ for any $FP$-injective left $R$-module $N$ and call $\text{fid}(M)$ the FI-injective dimension of $M$. If no such $n$ exists, set $\text{fid}(M) = \infty$.

Put $lfiD(R) = \sup\{\text{fid}(M) : M \text{ is any left } R\text{-module}\}$, and call $\text{fid}(R)$ the left FI-injective dimension of $R$. Similarly, we have $rfid(R)$. If $\text{fid}(M) = 0$, then $M$ is FI-injective module.

Remark 3.2 (1) We also use $\mathcal{FI}$-preenvelope to characterize FI-injective dimension, i.e., $\text{fid}(M) = \inf\{n : \text{there is an } \mathcal{FI}\text{-preenvelope resolution of the form } 0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{n-1} \rightarrow E^n \rightarrow 0 \text{ of } M\}$.

(2) It is clear that $\text{fid}(M) \leq \text{id}(M)$ for any left $R$-module $M$ and $lfiD(R) \leq lD(R)$ for any ring $R$. It is also easy to see that a ring $R$ is semihereditary if and only if $\text{fid}(M) = \text{id}(M)$ for all left $R$-module $M$ if and only if every FI-injective left $R$-module is injective.
Proposition 3.3 Let $R$ be a left coherent ring. Then the following are equivalent for any left $R$-module $M$ and an integer $n \geq 0$:

1. $\text{fid}(M) \leq n$.
2. $\text{Ext}_R^i(N, M) = 0$ for any $FP$-injective left $R$-module $N$ and $i \geq n + 1$.
3. Every $n$-th cosyzygy of $M$ is $FI$-injective.
4. There exists an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$, where each $E^i$ is $FI$-injective.

Proof: (1) $\Leftrightarrow$ (2) by definition.

(2) $\Leftrightarrow$ (3) We simply note that if $L^n$ is $n$-th cosyzygy of $M$, then $\text{Ext}_R^{i+n}(N, M) \cong \text{Ext}_R^i(N, L^n)$, so the result follows.

(1) $\Leftrightarrow$ (4) is straightforward.

Corollary 3.4 If $R$ is noetherian ring and $\text{id}(M) < \infty$, then $\text{fid}(M) = \text{id}(M)$.

Proof: $\text{fid}(M) \leq \text{id}(M)$ follows from Remark 3.2. Since injectives are $FI$-injectives. Now suppose $\text{fid}(M) = n$. Then $\text{Ext}_R^i(N, M) = 0$ for all $FP$-injectives $N$ and $i \geq n + 1$. But $\text{id}(M) < \infty$, so $\text{id}(M) \leq n$ by [7, Lemma 2.2].

Proposition 3.5 Let $0 \to M \to E^0 \to E^1 \to \cdots$ be a $FI$-injective resolution of an $R$-module $M$. Then the sequence $0 \to T \otimes_R M \to T \otimes_R E^0 \to T \otimes_R E^1 \to \cdots$ is exact for all right strongly $FI$-flat $R$-module $T$.

Proof: Let $K^i = \ker(E^i \to E^{i+1})$, $i \geq 1$ and $K^0 = M$. We consider the short exact sequence $0 \to K^i \to E^i \to K^{i+1} \to 0$. Then $0 \to T \otimes_R K^i \to T \otimes_R E^i \to T \otimes_R K^{i+1} \to 0$ is exact if and only if $0 \to (T \otimes_R K^{i+1})^+ \to (T \otimes_R E^i)^+ \to (T \otimes_R K^i)^+ \to 0$ is exact. But the latter is equivalent to $0 \to \text{Hom}_R(K^{i+1}, T^+) \to \text{Hom}_R(E^i, T^+) \to \text{Hom}_R(K^i, T^+) \to 0$ being exact. So the result follows since $T^+$ is strongly $FI$-injective by [8, Remark 2.2(2)].

Proposition 3.6 Let $R$ be a left coherent ring, $0 \to A \to B \to C \to 0$ an exact sequence of left $R$-modules. If two of $\text{fid}(A)$, $\text{fid}(B)$ and $\text{fid}(C)$ are finite, so is the third. Moreover,

1. $\text{fid}(B) \leq \sup\{\text{fid}(A), \text{fid}(C)\}$.
2. $\text{fid}(A) \leq \sup\{\text{fid}(B), \text{fid}(C) + 1\}$.
3. $\text{fid}(C) \leq \sup\{\text{fid}(B), \text{fid}(A) - 1\}$.

Proof: By Definition and Proposition 3.3.

Corollary 3.7 Let $R$ be a left coherent ring. Then the following hold:

1. If $0 \to A \to B \to C \to 0$ is an exact sequence of left $R$-modules, where $1 < \text{fid}(A) < \infty$ and $B$ is $FI$-injective, then $\text{fid}(C) = \text{fid}(A) - 1$.
2. $\text{rfid}(R) = n$ if and only if $\sup\{\text{fid}(I) : I$ is any left ideal of $R\} = n + 1$ for any integer $n \geq 0$. 
\textbf{Proof:} (1) is true by Proposition 3.6.
(2) For a left ideal $I$ of $R$, consider the exact sequence $0 \to I \to R \to R/I \to 0$. Then (2) follows from (1).

\textbf{Theorem 3.8} Let $R$ be a left coherent ring. Then the following are equivalent:

\begin{enumerate}
\item $lfiD(R)$
\item $\sup\{fid(M): M \text{ is any left } R\text{-module}\}$
\item $\sup\{pd(F): F \text{ is an FP-injective left } R\text{-module}\}$
\item $\sup\{fid(F): F \text{ is an FP-injective left } R\text{-module}\}$
\end{enumerate}

\textbf{Proof:} (1) $\Leftrightarrow$ (2) and (4) $\leq$ (2) are obvious.
(2) $\leq$ (3) Assume $\sup\{pd(F): F \text{ is an FP-injective left } R\text{-module}\} = M < \infty$. Let $M$ be any left $R$-module and $N$ any FP-injective left $R$-module. Since $pd(N) \leq M$, it follows that $\text{Ext}^{m+1}_R(N, M) = 0$. Hence $fid(M) \leq m$.

(2) $\leq$ (4) We may assume that $\sup\{fid(F): F \text{ is an FP-injective left } R\text{-module}\} = n < \infty$. Let $M$ be any left $R$-module. By Theorem 2.3, there is a short exact sequence $0 \to K \to F \to M \to 0$, where $F$ is FP-injective and $K$ is $FI$-injective. Thus $fid(M) \leq fid(F) \leq n$, as desired.

(3) $\leq$ (4) is easy.

\textbf{Corollary 3.9} Let $R$ be a left coherent ring. Then the following are equivalent for an integer $n \geq 0$:

\begin{enumerate}
\item $lfiD(R) \leq n$.
\item $pd(M) \leq n$ for all FP-injective left $R$-modules $M$.
\item $fid(M) \leq n$ for all FP-injective left $R$-modules $M$.
\item $pd(M) \leq n$ for all left $R$-modules $M$ that are both $FI$-injective and FP-injective and $lfiD(R) < \infty$.
\item $fid(M) \leq n$ for all projective left $R$-modules $M$, and $lfiD(R) < \infty$.
\item $\text{Ext}^{n+1}_R(N, M) = 0$ for all FP-injective left $R$-modules $M$, $N$ and $j \geq 1$.
\end{enumerate}

\textbf{Proof:} By Theorem 3.8, it suffices to show that (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3).

(4) $\Rightarrow$ (2) Let $M$ be any FP-injective left $R$-module. Since $lfiD(R) < \infty$, $fid(M) = m$ for a nonnegative integer $m$ by Theorem 3.8 (4). Note that every left $R$-module has a special $FI$-preenvelope by Theorem 2.3, then there exists an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{m-1} \to E^m \to 0$, where each $E^i$ is both $FI$-injective and FP-injective since $pd(E^i) \leq n$ by (4), $pd(M) \leq n$.

(5) $\Rightarrow$ (3) Let $M$ be any FP-injective left $R$-module. Since $lfiD(R) < \infty$, $pd(M) = m$ for all integer $m \geq 0$ by Theorem 3.8 (3). Hence $M$ admits an projective resolution $0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$. Note that $fid(P_i) \leq n$ for each $P_i$ by (5), so $fid(M) \leq n$ by Proposition 3.6.
Proposition 3.10 Let \( R \) be a left coherent ring. Then \( lfiD(R) \geq \sup\{pd(M) : M \) is a left \( R \)-module with \( id(M) < \infty \} \geq \sup\{pd(M) : M \) is an injective left \( R \)-module\}.

Proof: Suppose \( lfiD(R) = n \). Let \( M \) be a left \( R \)-module with \( id(M) = m < \infty \). Then we have an exact sequence \( 0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{m-1} \rightarrow E^m \rightarrow 0 \). Note that every \( E^i \) is \( FP \)-injective, so \( pd(E^i) \leq n \) by Theorem 3.8, whence \( pd(M) \leq n \), as desired. The second inequality is trivial.

References


Received: November, 2011