Rational Gottlieb Group of Function Spaces of Maps into an Even Sphere

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Abstract. Let $X$ be a simply-connected space and $f : X \to S^{2n}$ a mapping into the even sphere. We show that the dimension of the rational Gottlieb group of the universal cover $\tilde{\text{map}}(X, S^{2n}; f)$ of the function space $\text{map}(X, S^{2n}; f)$ is at least equal to the dimension of $\tilde{H}^*(X, \mathbb{Q})$, if the image of $\tilde{H}^*(f, \mathbb{Q})$ is in the annihilator of $\tilde{H}^*(X, \mathbb{Q})$.

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1. INTRODUCTION

An element $\alpha \in \pi_n(X)$ is called a Gottlieb element of $X$ if the map $(\alpha, \text{id}) : S^n \vee X \to X$ extends to $\tilde{\alpha} : S^n \times X \to X$ [7]. Gottlieb elements form a subgroup $G_n(X)$ of $\pi_n(X)$. Moreover $G_n(X)$ is the image of the connecting map of the long exact sequence of homotopy groups of the universal fibration $X \to B \text{aut}^\bullet X \to B \text{aut} X$ [8]. Here $\text{aut} X$ denotes the monoid of self homotopy equivalences of $X$ and $\text{aut}^\bullet X$ its submonoid of pointed self homotopy equivalences. Therefore, given a fibration $X \to E \to B$, the image of the connecting map $\delta : \pi_{n+1}(B) \to \pi_n(X)$ is contained in the Gottlieb group $G_n(X)$. Moreover, if $X_0$ denotes the rationalization of $X$, then $G_n(X) \otimes \mathbb{Q} \subset G_n(X_0)$ and equality holds if $X$ is a simply connected finite CW-complex. $G_n(X_0)$ is called the rational Gottlieb group of $X$. Useful descriptions of rational Gottlieb groups are given in [3, 14]. However little is known about rational Gottlieb groups of function spaces. In this paper we prove the following result.

Theorem. Let $f : X \to S^{2n}$ be a map such that $H^{2n}(f, \mathbb{Q}) = \alpha$ is in the annihilator of $\tilde{H}^*(X, \mathbb{Q})$. Then $\dim G_n(\tilde{\text{map}}(X, S^{2n}; f)_0) \geq \dim \tilde{H}_*(X, \mathbb{Q})$.

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2. Rational spherical fibrations

In the sequel we will rely on Sullivan and Quillen theories of rational homotopy, of which details can be found in [13, 11, 14, 4]. We recall here Sullivan and Quillen models of spaces. A Sullivan algebra is a commutative cochain algebra of the form \((\wedge V, d)\), where \(V = \bigoplus_{i \geq 2} V_i\), Moreover \(V = \bigcup_{k \geq 0} V(k)\), where \(V(0) \subset V(1)\ldots \) such that \(d(V(0)) = 0\) and \(d(V(k)) \subset \wedge V(k - 1)\). It is called minimal if \(dV \subset \wedge \geq 2 V\). For any commutative differential graded algebra \((A, d)\) of which the cohomology is connected and finite in each degree, there is a unique minimal Sullivan algebra \((\wedge V, d)\) equipped with a quasi-isomorphism \((\wedge V, d) \rightarrow (A, d)\).

A Sullivan model of a simply connected topological space \(X\) is a Sullivan algebra \((\wedge V, d)\) which algebraically models the rational homotopy type of \(X\). It is called minimal if \((\wedge V, d)\) is minimal.

A simply connected space \(X\) is called formal if there is a quasi-isomorphism \((\wedge V, d) \rightarrow H^*(X, \mathbb{Q})\). Examples of formal spaces include spheres, homogeneous spaces \(G/H\) of which \(\text{rank}(G) = \text{rank}(H)\) and Kähler manifolds.

A Lie model is a differential graded Lie algebra \((L, \delta)\) that determines the rational homotopy type of \(X\). It is a Quillen model if it is of the form \((L(V), \delta)\) where \(L(V)\) denotes the free graded Lie algebra on the graded vector space \(V = \bigoplus_{i \geq 1} V_i\), and it is called minimal if \(\delta V \subset \mathbb{L} \geq 2 V\). For a simply connected space \(X\), there exists a minimal Quillen model \((L(V), \delta)\). Moreover its homology is isomorphic to \(\pi_*(\Omega X) \otimes \mathbb{Q}\), the (rational) homotopy Lie algebra of \(X\) and \(V_n \cong H_{n+1}(X, \mathbb{Q})\). A map of spaces \(f : X \rightarrow Y\) induces a map of Quillen models which we denote abusively by \(f : (L(V), \delta) \rightarrow (L(W), \delta')\). A simply connected space \(X\) is called coformal if there is a quasi-isomorphism \((L(V), \delta) \rightarrow \pi_*(\Omega X) \otimes \mathbb{Q}\).

For a simply connected CW-complex, we denote by \(\text{aut}_1(X)\) the monoid of self homotopy equivalences of \(X\) that are homotopic to the identity and \(\text{aut}_1^*(X)\) its sub monoid of pointed self homotopy equivalences. There is a universal fibration \(X \rightarrow B\text{aut}_1^*(X) \rightarrow B\text{aut}_1(X)\). It classifies fibrations of which the fibre has the homotopy type of \(X\) and the base space is simply connected [2]. Such fibrations can be expressed in terms of Sullivan or Quillen models, hence they are called rational fibrations. Universal rational fibrations have been studied by Sullivan, Schlessinger-Stasheff, Tanré among others [13, 12, 14].

If \(X\) has the rational homotopy type of an even dimensional sphere \(S^{2n}\), then the space \(B\text{aut}_1(X)\) has the rational homotopy type of \(K(\mathbb{Q}, 4n)\) [?, 14, 5]. Consider the map \(\alpha : S^{4n} \rightarrow B\text{aut}_1(X)\) that corresponds to the generator of \(\pi_{4n}(B\text{aut}_1(X)) \otimes \mathbb{Q}\). The pullback along the universal fibration gives rise to a rational fibration \(S^{2n} \rightarrow E \rightarrow S^{4n}\). One can describe this fibration by the
means of the KS-extension
\[ \wedge((x_{4n}, x_{8n-1}), d) \rightarrow (\wedge(x_{4n}, x_{8n-1}, x_{2n}, x_{4n-1}), D) \rightarrow (\wedge(x_{2n}, x_{4n-1}), d'), \]
where \( D x_{2n} = 0, \ D x_{4n-1} = x_{2n}^2 - x_{4n} \) (subscripts indicate degrees). The minimal Sullivan model of \( E \) is \( (\wedge(x_{2n}, x_{8n-1}), d), \) with \( d x_{8n-1} = x_{2n}^4. \)

As \( E \) is formal, it has a Quillen model of the form \((\mathbb{L}(x_{2n-1}, x_{4n-1}, x_{6n-1}), \delta)\) where \( \delta x_{2n-1} = 0, \ \delta x_{4n-1} = [x_{2n-1}, x_{2n-1}] \) and \( \delta x_{6n-1} = [x_{2n-1}, x_{4n-1}] \). It is the only non trivial deformation of the Quillen minimal model of the product \( S^{4n} \times S^{2n} \) in the sense of Tanrè [14, chap.7]. Moreover, a non free model of \( E \) is given by \((\mathbb{L}(x) \times x \mathbb{L}(y), d')\), where \( \mathbb{L}(x) \times x \mathbb{L}(y) \) is isomorphic to the product Lie algebra \( \mathbb{L}(x) \times \mathbb{L}(y) \) and the differential is defined by \( d'x = 0 \) and \( d'y = [x, x] \).

A Lie model of the fibration \( S^{2n} \rightarrow E \xrightarrow{p} S^{4n} \) is then given by
\[
\begin{align*}
\begin{array}{c}
(\mathbb{L}(x), 0) \xrightarrow{i} (\mathbb{L}(x) \times x \mathbb{L}(y), d') \xrightarrow{p} (\mathbb{L}(y), 0) \\
\cong \xrightarrow{q} (\mathbb{L}(x_{2n-1}, x_{4n-1}, x_{6n-1}), \delta),
\end{array}
\end{align*}
\]

where \( q \) is the canonical quasi-isomorphism defined by \( q(x_{2n-1}) = x, \ q(x_{4n-1}) = y \) and \( q(x_{6n-1}) = 0 \). As observed in [14], the connecting map of the long homotopy exact sequence of the above fibration is not zero, indeed \( \delta : \pi_{4n}(S^{4n}) \otimes \mathbb{Q} \rightarrow \pi_{4n-1}(S^{2n}) \otimes \mathbb{Q} \) is an isomorphism.

3. Rational Gottlieb group of \( \tilde{\operatorname{map}}(X, S^{2n}) \).

For a map \( f : X \rightarrow Y \), we denote by \( \operatorname{map}(X, Y; f) \) the connected component of \( \operatorname{map}(X, Y) \) containing \( f \). Our aim is to compute the rational Gottlieb group of \( \tilde{\operatorname{map}}(X, S^{2n}; f) \), the universal cover of \( \operatorname{map}(X, S^{2n}; f) \), under additional assumptions on \( \tilde{H}^*(f, \mathbb{Q}) \). Given the above fibration \( S^{2n} \xrightarrow{i} E \xrightarrow{p} S^{4n} \) and a mapping \( f : X \rightarrow S^{2n} \), one gets a fibration between function spaces
\[
\begin{align*}
\mathcal{F} \xrightarrow{i} \operatorname{map}(X, E; i \circ f) \xrightarrow{\operatorname{map}(p)} \operatorname{map}(X, S^{4n}; c),
\end{align*}
\]
where \( c \) is the constant map.

**Lemma 1.** The fibre \( \mathcal{F} \) is rationally equivalent to \( \operatorname{map}(X, S^{2n}; f) \).

**Proof.** Let \( f : X \rightarrow S^{2n} \). If \( f \) is not trivial, we may assume that the induced map \( \phi : (\mathbb{L}(V), \delta) \rightarrow \mathbb{L}(x) \) verifies either \( \phi(v) = x \) or \( \phi(v) = [x, x] \). Hence \( \operatorname{map}(X, S^{2n}) \) has three rational path components.

Let \( g \in \mathcal{F} \). Then \( p \circ g \) is the constant map, therefore \( g \) factors into \( \bar{g} : X \rightarrow S^{2n} \). We assume first that \( \phi(v) = x \). A model of \( g \) is given by \( \gamma : (\mathbb{L}(V), \delta) \rightarrow \mathbb{L}(x) \times x \mathbb{L}(y), \delta) \) with \( \gamma(v) = x \). Therefore a model \( \varphi \) for \( \bar{g} \) will verify \( \varphi(x) = v \). Hence \( \bar{g} \) is rationally equivalent to \( f \).

If \( \phi(v) = [x, x] \), then \( i \circ f \) is rational trivial and the fibre of \( \operatorname{map}(p) : \operatorname{map}(X, E; c) \rightarrow \operatorname{map}(X, S^{4n}; c) \) is rationally homotopic to \( \operatorname{map}(X, S^{2n}; c) \).
Therefore $\text{map}(X, S^{2n}; f)$ and $\text{map}(X, S^{2n}; c)$ have same rational homotopy type [10].

Finally if $\phi(v) = 0$, then both $f$ and $\bar{g}$ are trivial and the fibre has the rational homotopy type of $\text{map}(X, S^{2n}; c)$. In all three cases, the fibre $\mathcal{F}$ has the rational homotopy type of $\text{map}(X, S^{2n}; f)$.

Let $(\mathbb{L}(V), \delta)$ be the Quillen minimal model of $X$ and $(T(V), d)$ its enveloping algebra.

It is known that there is an acyclic differential $T(V)$-module of the form $(T(V) \otimes (Q \oplus sV), D)$ [1, 9]. The differential $D$ is given by

$$D(v \otimes 1) = dv \otimes 1, \quad Dsv = v \otimes 1 - S(dv \otimes 1),$$

where $S$ is the $Q$-graded vector space map (of degree 1) defined by

$$S(v \otimes 1) = 1 \otimes sv, \quad S(1 \otimes (Q \oplus sV)) = 0,$$

and

$$S(ax \otimes 1) = (-1)^{|a|}aS(x \otimes 1), \quad \forall a \in TV, \quad |x| > 0.$$

Recall that if $A$ is a differential graded algebra and $(M, d_M)$ and $(N, d_N)$ are differential graded $A$-modules, $\text{Hom}_A(M, N)$ is a differential vector space of which the differential is defined by

$$Df = d_N \circ f - (-1)^{|f|}f \circ d_M.$$

Let $f : X \to Y$ be a map between spaces, $\phi : (\mathbb{L}(V), \delta) \to (\mathbb{L}(W), \delta')$ its Quillen model and $U\phi : TV \to TW$ the induced mapping between enveloping algebras. The adjoint action of $TW$ on $\mathbb{L}(W)$ combined with $U\phi$ induces a $T(V)$-module structure on $\mathbb{L}(W)$. For $n \geq 1$, it was shown that there is an isomorphism [6]

$$\pi_n(\Omega \text{map}(X, Y; f)) \otimes \mathbb{Q} \cong \text{Ext}^n_{TV}(Q, \mathbb{L}(W)) \cong H_n((\text{Hom}_{TV}(TV \otimes (Q \oplus sV), \mathbb{L}(W)), D)).$$

Let $P = TV \otimes (\mathbb{Q} \oplus sV)$. The complex $C = \text{Hom}_{TV}(P, \mathbb{L}(W))$ is not positively graded. If we put

$$\tilde{C}_i = \begin{cases} C_i & \text{for } i > 1 \\ Z(C)_1 & \text{for } i = 1, \end{cases}$$

then the complex $\tilde{C}$ computes the rational homotopy groups of the universal cover $\widetilde{\text{map}}(X, Y; f)$ of $\text{map}(X, Y; f)$.

The long exact sequence of the fibration

$$\text{map}(X, S^{2n}; f) \to \text{map}(X, E; i \circ f) \to \text{map}(X, S^{4n}; c)$$

can be computed from the exact sequence

(3)$$\text{Hom}(Q \oplus sV, \mathbb{L}(x)) \xrightarrow{1} \text{Hom}(Q \oplus sV, \mathbb{L}(x) \times_{\alpha} \mathbb{L}(y)) \xrightarrow{\psi} \text{Hom}(Q \oplus sV, \mathbb{L}(y)).$$

As the differential on $\text{Hom}(Q \oplus sV, \mathbb{L}(y))$ is zero, then

$$\pi_*(\widetilde{\text{map}}(X, S^{4n}; c) \otimes \mathbb{Q} \cong \text{Hom}(Q \oplus sV, \mathbb{L}(y)).$$
The long exact sequence of the fibration (2) becomes

\[ \rightarrow \operatorname{Ext}_{TV}(Q, L(x) \times_\alpha L(y)) \overset{H(\psi)}{\rightarrow} \operatorname{Hom}(Q \oplus sV, L(y)) \overset{\delta}{\rightarrow} \operatorname{Ext}_{TV}(Q, L(x)) \rightarrow \]

We have the following result.

**Theorem 2.** Let \( f : X \rightarrow S^{2n} \) be a map such that \( H^{2n}(f, \mathbb{Q}) = \alpha \) is in the annihilator of \( \bar{H}^*(X, \mathbb{Q}) \). Then \( \dim G_*\operatorname{map}(X, S^{2n}; f)_0 \geq \dim \bar{H}_*(X, \mathbb{Q}) \).

**Proof.** We use the connecting map of the long exact sequence (4).

We first consider the case where \( \theta(1) = y \) and \( \theta(sV) = 0 \). There is \( \bar{\theta} \in \operatorname{Hom}(\mathbb{Q} \oplus sV, L(x) \times_\alpha L(y)) \) such that \( \psi(\bar{\theta}) = \theta \). One can choose \( \bar{\theta}(1) = y \) and zero on \( sV \). As \( (D\bar{\theta})(1) = [x, x] \), hence \( \delta \bar{\theta} \in \operatorname{Ext}_{TV}(\mathbb{Q} \oplus sV, L(x)) \) is represented by a cocycle \( \gamma \in \operatorname{Hom}(\mathbb{Q} \oplus sV, L(x)) \) such that \( \gamma(1) = [x, x] \) and \( \gamma(sV) = 0 \). This cocycle cannot be a coboundary as \( [x, x] \) is a non zero homology class in \( L(x) \). Consequently \( \delta(\theta) \in \operatorname{Ext}_{TV}(\mathbb{Q}, L(x)) \) is not zero.

Let \( \{v_1, v_2, \ldots\} \) be a basis of \( V \). Consider now \( \theta \in \operatorname{Hom}(\mathbb{Q} \oplus sV, L(y)) \) such that \( \theta(sv_k) = y \) and zero otherwise. In the same way, we define \( \bar{\theta} \in \operatorname{Hom}(\mathbb{Q} \oplus sV, L(x) \times_\alpha L(y)) \) by \( \bar{\theta}(sv_k) = y \) and zero otherwise. For any \( w \in \{v_1, v_2, \ldots\} \),

\[
(D\bar{\theta})(sw) = d\bar{\theta}(sw) - (-1)^{|\theta|}\bar{\theta}(w \otimes 1 - S(dw \otimes 1)) = d\bar{\theta}(sw) - (-1)^{|\theta||sw|}[\phi(w), \bar{\theta}(1)] + (-1)^{|\theta|}\bar{\theta}(S(dw \otimes 1)) = d\bar{\theta}(sw) + (-1)^{|\theta|}\bar{\theta}(S(dw \otimes 1)) \quad \text{(as } \bar{\theta}(1) = 1).}
\]

As \( dv_k \) is a polynomial in variables of degree less that \( |v_k| \), we deduce that \( \bar{\theta}(S(dv_k \otimes 1)) = 0 \), therefore \( (D\bar{\theta})(sv_k) = [x, x] \). Moreover for \( v_j \neq v_k \), \( (D\bar{\theta})(sv_j) = (-1)^{|\theta||sv_j|}\bar{\theta}(S(dv_j \otimes 1)) \). Therefore we define \( \bar{\theta} \in \operatorname{Hom}(\mathbb{Q} \oplus sV, L(x)) \) by \( \bar{\theta}(1) = 0 \), \( \bar{\theta}(sv_k) = [x, x] \) and \( \bar{\theta}(sv_j) = (-1)^{|\theta||sv_j|}\bar{\theta}(S(dv_j \otimes 1)) \) for \( v_j \neq v_k \).

We now show that \( \bar{\theta} \) cannot be a coboundary. On the contrary assume that there exists \( \varphi \in \operatorname{Hom}(\mathbb{Q} \oplus sV, L(x)) \) such that \( D\varphi = \bar{\theta} \). Let \( v \in V \) be the dual of \( \alpha \in H^{2n}(X, \mathbb{Q}) \). We suppose that \( v_k \neq v \) so that \( \phi(v_k) = 0 \) or \( \phi(v_k) = r[x, x] \).

Therefore

\[
(D\varphi)(sv_k) = \delta\varphi(sv_k) - (-1)^{|\varphi|}\varphi(sv_k \otimes 1 - S(dv_k \otimes 1)) = (-1)^{|\varphi||sv_k|}[\phi(sv_k), \varphi(1)] + (-1)^{|\varphi|}\varphi(S(dv_k \otimes 1)) = (-1)^{|\varphi||sv_k|}\varphi(S(dv_k \otimes 1)).
\]

We write \( dv_k = d_2v_k + d_3v_k + \ldots \), where \( d_i v_k \in T^i(V) \). Therefore \( S(dv_k \otimes 1) = S(d_2v_k \otimes 1) + S(d_3v_k \otimes 1) + \ldots \). Moreover \( \varphi(S(dv_k \otimes 1)) \in L^{\geq 3}(x) \) for \( i \geq 3 \). But the polynomial \( dv_k \) does not contain \( v \), as the dual of \( v \) annihilates \( \bar{H}^*(X, \mathbb{Q}) \). Therefore \( \varphi(S(dv_k \otimes 1)) = 0 \) as well. We conclude that each element in \( \{1, v_1, v_2, \ldots\} \cong \mathbb{H}^*(X, \mathbb{Q}) \), distinct from \( v \), gives rise to a non zero Gottlieb
element $\tilde{\theta}_k$ in $\pi_*(\Omega \tilde{\text{map}}(X, S^{2n}; f)) \otimes \mathbb{Q} \cong \text{Ext}_{TV}(\mathbb{Q}, \mathbb{L}(x))$. By construction the $\{\tilde{\theta}_k\}$ are linearly independent.

From the above theorem, we can immediately deduce the following result.

**Corollary 3.** If $f : S^{2n} \vee X \to S^{2n}$ is the map that consists in collapsing $X$ onto a point, then the rational Gottlieb group of $\tilde{\text{map}}(S^{2n} \vee X, S^{2n}; f) \geq \dim H^*(X, \mathbb{Q})$.

**Example 4.** We consider the map $f : \mathbb{C}P(2) \to S^4$ of which the minimal model is given by

$$\phi : (\wedge (x_4, x_7), d) \to (\wedge (x_2, x_5), d),$$

where $dx_4 = 0$, $dx_7 = x_4^2$, $dx_2 = 0$, $dx_5 = x_2^3$ and $\phi(x_4) = x_2^2$, $\phi(x_7) = x_2x_5$. As the image of $\tilde{H}^*(f, \mathbb{Q})$ annihilates $\tilde{H}^*(\mathbb{C}P(2), \mathbb{Q})$, we deduce that the rational Gottlieb group of $\tilde{\text{map}}(\mathbb{C}P(2), S^4; f)$ is of dimension at least 2.

**References**


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