Some Finiteness Properties of Generalized
Graded Local Cohomology Modules

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Abstract

Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, m_0)$ and let $M$ and $N$ be finitely generated graded $R$–modules. Let $i, j \in \mathbb{N}_0$. In this paper we will study Artinianess of $\Gamma_{m_0 R}^i(M, N)$, $H^i_{m_0 R}(H^j_{R_+}(M, N)/m_0 H^j_{R_+}(M, N))$, $H^i_{R_+}(M, H^j_{m_0 R}(N))$, $H^j_{m_0 R}(M, H^i_{R_+}(N))$, where $R_+$ denotes the irrelevant ideal of $R$.

Keywords: Graded local cohomology, Generalized graded local cohomology, Artinian module

1 Introduction

Throughout this paper, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, m_0)$. So that $R_0$ is a Noetherian ring and there are finitely many elements $l_1, l_2, \ldots, l_r \in R_1$ such that $R = R_0[l_1, l_2, \ldots, l_r]$. Let $R_+ = \bigoplus_{n \in \mathbb{N}} R_n$ denotes the irrelevant ideal of $R$ and let $m = m_0 \oplus R_+$ denote the graded maximal ideal of $R$. Finally let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $N = \bigoplus_{n \in \mathbb{Z}} N_n$ be finitely generated
graded $R$–modules. Herzog introduced a generalization of local cohomology so called generalized local cohomology, denoted by $H^i_{R^+}(M, N)$ which is isomorphic to $\lim_{n} \text{Ext}^i_R(M/R^+M, N)$. We note that if $M = R$, then $H^i_{R^+}(R, N) = H^i_{R^+}(N)$ is the usual local cohomology. As is well known, the finiteness of the local cohomology modules have an important role in commutative algebra and algebraic geometry. Many of mathematicians work on finiteness of local cohomology. One of approaches in finiteness is Artinianess. Authors, in [1], [2], [7] and [8] studied Artinianess of the graded modules $\Gamma_{m_0R}(H^i_{R^+}(N)), H^i_{R^+}(N)/m_0H^i_{R^+}(N)$ and $H^1_{m_0R}(H^i_{R^+}(M))$. In the recent paper we will study the Artinianess of generalized graded local cohomology. Here, we briefly mention some our results which have been proved in this paper. Let $f$ be the least non-negative integer such that the graded module $H^i_{R^+}(M, N)$ is not finitely generated. We prove that $\Gamma_{m_0R}(H^i_{R^+}(M, N))$ is Artinian for each $i \leq f$. We also prove that if $M$ is of finite projective dimension with $pd_RM = n$ and $c = c_{R^+}(N)$ is the largest non-negative integer $i$ such that $H^i_{R^+}(N) \neq 0$, then $H^{i+c}_{R^+}(M, N)/m_0H^{i+c}_{R^+}(M, N)$ is Artinian. In particular if $c_{R^+}(M, N)$ is the largest non-negative integer $i$ such that $H^i_{R^+}(M, N) \neq 0$, then $c_{R^+}(M, N) \leq pd_RM + c_{R^+}(N)$. We prove a similar result for invariant $a_{R^+}(N)$ which shows the largest non-negative integer $i$ such that $H^i_{R^+}(N)$ is not Artinian. Moreover, we show that if $H^i_{R^+}(M, N)$ is $R^+-$cofinite, then $\Gamma_{m_0R}(H^i_{R^+}(M, N))$ is Artinian. Finally we study Artinianess of generalized local cohomology when $\text{dim}(R_0) \leq 1$. In this case, we prove that $H^i_{R^+}(M, N)/m_0H^i_{R^+}(M, N), \Gamma_{m_0R}(H^i_{R^+}(M, N)), H^1_{m_0R}(H^i_{R^+}(M, N))$, $H^j_{R^+}(M, H^1_{m_0R}(N))$ and $H^j_{m_0R}(M, H^1_{R^+}(N))$ are Artinian for all $i, j \in N_0$.

2 The Results

It should be noted that local flat morphism of local Noetherian rings is faithfully flat. So, if $R'_0$ is flat over $R_0$ and $m'_0 = m_0R'_0$, then $R'_0$ is faithfully flat over $R_0$. Moreover, it follows from [4, Theorem 1] that if $(R'_0, m'_0)$ is a faithfully flat local $R_0$-algebra, then $A$ is a graded Artinian $R$-module if and only if $A' := R'_0 \otimes_{R_0} A$ is a graded Artinian module over $R' := R'_0 \otimes_{R_0} R$. In view of this argument, we have the following proposition.

Proposition 2.1. Let $f := f_{R^+}(M, N)$ be the least non-negative integer such that $H^i_{R^+}(M, N)$ is not finitely generated. Then $\Gamma_{m_0R}(H^i_{R^+}(M, N))$ is Artinian for each $i \leq f$.

Proof. We prove the assertion by induction on $i$. If $i = 0$, then $\Gamma_R(M, N)$ is
finitely generated and hence $\Gamma_{m_0 R}(\Gamma_{R+}(M, N))$ is finitely generated. Thus there exists some positive integer $n$ such that $m^n \Gamma_{m_0 R}(\Gamma_{R+}(M, N)) = 0$ and so the result follows in this case. Now, suppose inductively, that the result has been proved for all values smaller than or equal to $i$ with $i < f$ and we prove it for $i+1$, where $i+1 \leq f$.

Consider the exact sequence $0 \to \Gamma_{R+}(N) \to N \to N/\Gamma_{R+}(N) \to 0$. Application of the functor $H^i_{R+}(M, -)$ to the above exact sequence induces the following exact sequence:

\[
H^i_{R+}(M, \Gamma_{R+}(N)) \to H^i_{R+}(M, N) \xrightarrow{\varphi} H^i_{R+}(M, N/\Gamma_{R+}(N)) \to H^{i+1}_R(M, \Gamma_{R+}(N)) \cdots (*).
\]

Set $K_1 = \ker \varphi$ and $K_2 = \coker \varphi$. We note that for each $i$, there is an isomorphism $H^i_{R+}(M, \Gamma_{R+}(N)) \cong \text{Ext}^i(M, \Gamma_{R+}(N))$ and then this module is an $R_+-$torsion finitely generated graded $R-$module. Therefore $K_1$ and $K_2$ are $R_+-$torsion finitely generated graded $R-$module. Consider $\Gamma_{m_0 R}(K_1)$ and $\Gamma_{m_0 R}(K_2)$. By the previous argument these modules are finitely generated and $R_+-$torsion and then there exist some positive integer $n$ such that $m^n \Gamma_{m_0 R}(K_1) = m^n \Gamma_{m_0 R}(K_2) = 0$ and this implies that $\Gamma_{m_0 R}(K_1)$ and $\Gamma_{m_0 R}(K_2)$ are Artinian. Now, in view of the sequence (*) we can conclude that $\Gamma_{m_0}(H^i_{R+}(M, N))$ is Artinian if and only if $\Gamma_{m_0}(\tilde{H}^i_{R+}(M, N/\Gamma_{R+}(N)))$ is Artinian. So we may assume that $\Gamma_{R_+}(N) = 0$. Let $X$ be an indeterminate and let $R_0' = R_0[X]/m_0 R_0[X]$, $m'_0 = m_0 R_0'$, $R' = R_0' \otimes_{R_0} R$, $M' = R'_0 \otimes_{R_0} M$ and $N' = R_0' \otimes_{R_0} N$. Then by the flat base change property of local cohomology, for each $i$ we have $R'_0 \otimes_{R_0} \Gamma_{m_0 R}(H^i_{R+}(M, N)) \cong \Gamma_{m_0 R_0'}(H^i_{R_0' \otimes_{R_0} R_+}(R_0' \otimes_{R_0} M, R'_0 \otimes_{R_0} N))$ and using the above argument $\Gamma_{m_0}(H^i_{R+}(M, N))$ is Artinian if and only if $\Gamma_{m_0 R_0'}(H^i_{R_0' \otimes_{R_0} R_+}(R_0' \otimes_{R_0} M, R'_0 \otimes_{R_0} N))$ is Artinian. So we may assume that the residue field $k$ of $R_0$ is infinite. As $\Gamma_{R_+}(N) = 0$, there exists an element $x \in R_1$ which is a non-zero divisor with respect to $N$ and so there is an exact sequence

\[
0 \to N(-1) \xrightarrow{\varphi} N \to N/xN \to 0
\]

of graded $R-$modules. Application of the functor $H^i_{R_+}(M, -)$ to this exact sequence induces the exact sequence

\[
\ldots \xrightarrow{\varphi} H^i_{R_+}(M, N) \xrightarrow{\varphi} H^i_{R_+}(M, N/xN) \to H^i_{R_+}(-1)(M, N) \xrightarrow{\varphi} H^{i+1}_{R_+}(M, N).
\]

If we consider $U^i := H^i_{R_+}(M, N)/xH^i_{R_+}(M, N)$, then we have the exact sequence

\[
0 \to U^i \to H^i_{R_+}(M, N/xN) \to (0 : H^i_{R_+}(M, N))(-1) \to 0.
\]

Applying the functor $\Gamma_{m_0 R}(\cdot)$ to this sequence, we get the following exact sequence

\[
\Gamma_{m_0 R}(U^i) \cong H^i_{m_0 R}(U^i).
\]

We note that $U^i$ is finitely generated and $R_+-$torsion and so $H^i_{m_0 R}(U^i)$ is Artinian. On the other hand, one can easily show that $f(M, N/xN) \geq f(M, N) - 1$ and so $i + 1 \leq f_{R_+}(M, N)$ implies that $i \leq f_{R_+}(M, N) - 1 \leq f(M, N/xN)$; and hence by induction hypothesis $\Gamma_{m_0 R}(H^i_{R_+}(M, N/xN))$ is Artinian. Therefore, by
the sequence \((**\))\), the module \(\Gamma_{m_0}(0 : H_{R_+}^i(M,N)(-1) \to x) = (0 : \Gamma_{m_0}(H_{R_+}^i(M,N))(-1) \to x)\) is Artinian. Now, since \(\Gamma_{m_0}(H_{R_+}^i(M,N))\) is \(x\)-torsion, using Melkersson’s Lemma this module is Artinian.

**Lemma 2.2.** Let \(0 \to M_1 \to M \to M_2 \to 0\) be a short exact sequence of finitely generated graded \(R\)-modules. Then for any \(R\)-module \(N\), there is the following long exact sequence
\[
\cdots \to H_{R_+}^{i-1}(M_1,N) \to H_{R_+}^i(M_2,N) \to H_{R_+}^i(M,N) \to H_{R_+}^i(M_1,N) \to \cdots.
\]

**Proof.** Let \(I := 0 \to I_0 \to I_1 \to I_2 \to \cdots\) be an injective resolution of \(N\). We note that for any finitely generated \(R\)-module \(M\) and any \(i\), there is an isomorphism \(H_{R_+}^i(M,N) \cong H^i(\text{Hom}_R(M,\Gamma_{R_+}(I)))\) and by the basic properties of torsion functor each \(\Gamma_{R_+}(I)\) in \(\Gamma_{R_+}(I)\) is injective. Thus there is an exact sequence of complexes \(0 \to \text{Hom}_R(M_2,\Gamma_{R_+}(I)) \to \text{Hom}_R(M,\Gamma_{R_+}(I)) \to \text{Hom}_R(M_1,\Gamma_{R_+}(I)) \to 0\). Now, by using a basic theorem in homology theory, there is the following long exact sequence of \(R\)-modules \(\cdots \to H^{i-1}(\text{Hom}_R(M_1,\Gamma_{R_+}(I))) \to H^i(\text{Hom}_R(M_2,\Gamma_{R_+}(I))) \to H^i(\text{Hom}_R(M_1,\Gamma_{R_+}(I))) \to \cdots\) and this completes the proof.

**Theorem 2.3.** Let \(M\) be of finite projective dimension with \(pd_R(M) = n\) and \(c := c_{R_+}(N)\) be the largest positive integer \(i\) such that \(H_{R_+}^i(N)\) is not zero. Then the following conditions hold.

(i) The graded module \(H_{R_+}^{n+c}(M,N)/m_0H_{R_+}^{n+c}(M,N)\) is Artinian.

(ii) If \(c_{R_+}(M,N)\) is the largest positive integer \(i\) such that \(H_{R_+}^i(M,N)\) is not zero, then \(c_{R_+}(M,N) \leq pd_R(M) + c_{R_+}(N) = n + c\).

**Proof.** (i) We proceed by induction on \(pd_R(M) = n\). If \(n = 0\), then the result is clear by [7, Theorem 2.1]. Now, suppose inductively that the result has been proved for all values smaller than \(n > 0\) and so we prove this for \(n\). Since \(pd_R(M) = n\), there exists a positive integer \(t\) and an exact sequence of graded modules \(0 \to K \to R^t \to M \to 0\) such that \(pd_R(K) = n - 1\). In view of Lemma 2.2, if we apply the functor \(H_{R_+}^{n+c}(-,N)\) to this exact sequence, we have the following exact sequence
\[
H_{R_+}^{n+c-1}(K,N) \to H_{R_+}^{n+c}(M,N) \to H_{R_+}^{n+c}(R^t,N).
\]
We note that \(n + c > c = c_{R_+}(N)\) and so \(H_{R_+}^{n+c}(N) = 0\). Now, application of the functor \(R_0/m_0 \otimes_R -\) to the above exact sequence induces the following epimorphisms \(H_{R_+}^{n+c-1}(K,N)/m_0H_{R_+}^{n+c-1}(K,N) \to H_{R_+}^{n+c}(M,N)/m_0H_{R_+}^{n+c}(M,N) \to 0\). By using induction hypothesis, the graded mod-
u l e $H_{R+}^{n+c-1}(K, N)/m_0 H_{R+}^{n+c-1}(K, N)$ is Artinian. Thus the result follows by the above epimorphism.

(ii) In this part, similar to (i), we can apply an easy induction on $pd_R(M) = n$.

**Theorem 2.4.** Let $M$ be of finite projective dimension and let $a_{R+}(M, N)$ be the largest non-negative integer $i$ such that $H_{R+}^i(N)$ is not Artinian. Then we have the following conditions.

(i) $a_{R+}(M, N) \leq pd_R(M) + a_{R+}(N)$, where $a_{R+}(N)$ is the largest non-negative integer $i$ such that $H_{R+}^i(N)$ is not Artinian.

(ii) $H_{R+}^{n+n}(M, N)/m_0 H_{R+}^{n+n}(M, N)$ is Artinian, where $a = a_{R+}(N)$ and $pd_R(M) = n$.

**Proof.** (i) Let $pd_R(M) = n$ and $a = a_{R+}(N)$. We prove the assertion by induction on $pd_R(M) = n$. If $n = 0$, the result is clear. Suppose, inductively that $n > 0$ and the result has been proved for all values smaller than $n$ and so we prove it for $n$. As $pd_R(M) = n$, there exists a positive integer $t$ and an exact sequence $0 \rightarrow M_1 \rightarrow R^t \rightarrow M \rightarrow 0$ of $R$-modules such that $pd_R(M_1) = n - 1$. In view of Lemma 2.2, if we apply the functor $H_{R+}^i(-, N)$ to the above exact sequence, we get the following exact sequence of $R$-modules $H_{R+}^{i-1}(M_1, N) \rightarrow H_{R+}^i(M, N) \rightarrow H_{R+}^i(N)$\!. Now, consider $i > a + n$. We note that $i - 1 > a + n - 1 = pd_R(M_1) + a_{R+}(N)$ and $i > a + n > a$. Thus, using induction hypothesis, $H_{R+}^{i-1}(M_1, N)$ and $H_{R+}^i(M, N)$ are Artinian. Now, in view of Lemma 2.2, the result follows easily.

(ii) To prove this part, we again proceed by induction on $pd_R(M) = n$ . If $n = 0$, then the result follows by [8, Theorem 2.3]. Now suppose, inductively that $n > 0$ and the result has been proved for all values smaller than $n$ and so we prove it for $n$. By a similar proof which mentioned in (i), there exists an exact sequence of $R$-modules $H_{R+}^{n+n-1}(M_1, N) \rightarrow H_{R+}^{n+n}(M, N) \rightarrow H_{R+}^n(N)$\!. Consider $X = Im\alpha$ and $Y = Im\beta$. Since $a + n > a$, the module $Y/m_0Y$ is Artinian. On the other hand, by using induction hypothesis, one can easily see that $X/m_0X$ is Artinian. Now, the result follows easily.

**Lemma 2.5.** Let $i \in N_0$ and $H_{R+}^i(M, N)$ is $R_+-$cofinite. Then $\Gamma_{m_0}(H_{R+}^i(M, N))$ is Artinian.

**Proof.** Since $H_{R+}^i(M, N)$ is $R_+-$cofinite, $Hom_R(R/R_+, H_{R+}^i(M, N))$ is finitely generated and $R_+-$torsion. Thus $\Gamma_{m_0}(Hom_R(R/R_+, H_{R+}^i(M, N)) \cong \Gamma_{m_0}(R/R_+, H_{R+}^i(M, N)) = (0 : m_0 H_{R+}^i(M, N) R_+))$ is finitely generated and $m-$torsion. It implies that the last term is Artinian. On the other hand, since $\Gamma_{m_0}(H_{R+}^i(M, N))$ is $R_+-$torsion, it is Artinian.

**Proposition 2.6.** Let $M$ be of finite projective dimension and let $R_+$ be a principal
graded ideal of \( R \). Then \( \Gamma_{m_0}(H^i_{R^+}(M, N)) \) is Artinian for each \( i \).

Proof. As \( R^+ \) is principal, using [3, Theorem 2.8], \( H^i_{R^+}(M, N) \) is \( R^+ \)-cofinite for each \( i \). Now, the assertion follows by the previous lemma.

**Lemma 2.7.** Let \( M \) and \( N \) be finitely generated graded \( R \)-modules and \( N \) be \( m_0 \)-torsion. Then \( H^i_{R^+}(M, N) \) is Artinian for all \( i \).

Proof. Since \( N \) is \( m_0 \)-torsion, there exists an injective resolution \( I := 0 \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \ldots \) such that each \( I^i \) is \( m_0 \)-torsion. By the definition of generalized local cohomology there are the following isomorphisms \( H^i_{R^+}(M, N) \cong H^i((\Gamma_{m_0}R)(I)) \cong H^i((\Gamma_{m_0}R)(N)) \). By the basic properties of generalized local cohomology, the last term is Artinian; and hence the result follows.

**Proposition 2.8.** Let \( \dim R_0 \leq 1 \). Then for every \( i \in N_0 \), the module \( H^i_{R^+}(M, N) / m_0 H^i_{R^+}(M, N) \) is Artinian.

Proof. If \( \dim R_0 = 0 \), then \( N \) is \( m_0 \)-torsion and so in view of Lemma 2.7, the graded module \( H^i_{R^+}(M, N) \) is Artinian for each \( i \). By using [1, Lemma 2.2] we can get the assertion. Now, suppose that \( \dim R_0 = 1 \) and consider the short exact sequence \( 0 \rightarrow \Gamma_{m_0}R(N) \rightarrow N \rightarrow N/\Gamma_{m_0}R(N) \rightarrow 0 \) of graded \( R \)-modules. Application of the functor \( H^i_{R^+}(M, -) \) to this exact sequence induces the following exact sequence \( H^i_{R^+}(M, \Gamma_{m_0}R(N)) \rightarrow H^i_{R^+}(M, N) \rightarrow H^i_{R^+}(M, N/\Gamma_{m_0}R(N)) \rightarrow H^{i+1}_{R^+}(M, \Gamma_{m_0}R(N)) \). In view of Lemma 2.7 and using [1, Lemma 2.2], one can easily show that for each \( i \), the module \( R_0/m_0 \otimes_{R_0} H^i_{R^+}(M, N) \) is Artinian if and only if \( R_0/m_0 \otimes_{R_0} H^{i+1}_{R^+}(M, N/\Gamma_{m_0}(N)) \) is Artinian. So we can assume that \( \Gamma_{m_0}R(N) = 0 \). Now, this fact implies that there exists an element \( x \in m_0 \) which is a non-zero divisor of \( N \) and then there exists a short exact sequence \( 0 \rightarrow N \rightarrow N/xF \rightarrow 0 \) of graded \( R \)-modules. Application of the functor \( H^i_{R^+}(M, -) \) to the above exact sequence induces the following exact sequence \( H^i_{R^+}(M, N) \rightarrow H^i_{R^+}(M, N) \rightarrow H^i_{R^+}(M, N/xF) \rightarrow H^{i+1}_{R^+}(M, N) \). Since \( R_0 \) is of dimension one, \( N/xF \) is \( m_0 \)-torsion and so Lemma 2.7 implies that \( H^i_{R^+}(M, N/xF) \) is Artinian for each \( i \). Now, using [1, Lemma 2.2], \( R_0/m_0 \otimes_{R_0} H^i_{R^+}(M, N) \) is Artinian. On the other hand, application of the functor \( R_0/m_0 \otimes_{R_0} - \) to the above long exact sequence implies the following isomorphism \( R_0/m_0 \otimes_{R_0} H^i_{R^+}(M, N) \cong R_0/m_0 \otimes_{R_0} H^i_{R^+}(M, N) \otimes H^i_{R^+}(M, N) \) and hence the assertion follows.

**Proposition 2.9.** Let \( \dim R_0 \leq 1 \). Then we have the following conditions.

(i) The graded \( R \)-module \( \Gamma_{m_0}R(H^i_{R^+}(M, N)) \) is Artinian for each \( i \in N_0 \).
(ii) The graded $R$–module $H^1_{m_0R}(H^i_{R^+}(M, N))$ is Artinian for each $i \in N_0$.

Proof. If $\dim R_0 = 0$, then $R_0$ is Artinian. In this case any finitely generated $R$–module is Artinian and then for each $i$, the module $H^i_R(M, N)$ is Artinian. Thus $\Gamma_{m_0R}(H^i_{R^+}(M, N))$ is Artinian and $H^1_{m_0}(H^i_{R^+}(M, N)) = 0$ for each $i$ and so (i) and (ii) are clear in this case. Now, assume that $\dim R_0 = 1$.

(i) By applying the functor $H^i_{R^+}(M, -)$ to the exact sequence $0 \to \Gamma_{m_0R}(N) \to N \to N/\Gamma_{m_0R}(N) \to 0$ and applying the functor $\Gamma_{m_0R}(-)$ to the induced exact sequence, we can conclude that $\Gamma_{m_0R}(H^i_{R^+}(M, N))$ is Artinian if and only if $\Gamma_{m_0R}(H^i_{R^+}(M, N/\Gamma_{m_0R}(N)))$ is Artinian and so we may assume that $\Gamma_{m_0R}(N) = 0$. Now, let $x \in m_0$ be a non-zero divisor of $N$. Then there is an exact sequence $0 \to N \to N/xN \to 0$ of graded $R$–modules. Application of the functor $H^i_{R^+}(M, -)$ to this sequence induces the following exact sequence $H^i_{R^+}(M, N/xN) \to H^i_{R^+}(M, N) \to H^i_{R^+}(M, N)$. We note that $N/xN$ is $m_0$–torsion and so by Lemma 2.7, $H^i_{R^+}(M, N/xN)$ is Artinian and then $(0 : H^i_{R^+}(M, N) x)$ is Artinian. This implies that $\Gamma_{m_0}(0 : H^i_{R^+}(M, N) x) = (0 : r_{m_0R}(H^i_{R^+}(M, N)) x)$ is Artinian. Now, since $\Gamma_{m_0R}(H^i_{R^+}(M, N))$ is $x$–torsion, by using Melkesson’s Lemma, it is Artinian.

(ii) We proceed the assertion by induction on $i$. If $i = 0$, then $H^1_{m_0R}(\Gamma_{R^+}(M, N)) = H^1_{m_0}(\Gamma_{R^+}(M, N))$. We note that the last term is Artinian because $\Gamma_{R^+}(M, N)$ is finitely generated. Suppose, inductively that the result has been proved for all values smaller than $i$ and so we prove it for $i$. Let $y \in m_0$ be a system of parameters of $m_0$. As $M$ is finitely generated, for some positive integer $t$ there exists a short exact sequence $0 \to K \to R^t \to M \to 0$ of $R$–modules. In view of Lemma 2.2, if we apply the functor $H^i_{R^+}(-, N)$ to the above exact sequence, we get the following exact sequence $H^{i-1}_{R^+}(K, N) \to H^i_{R^+}(R^t, N) \to H^i_{R^+}(M, N) \to H^i_{R^+}(N/xN)$. Consider $A := \text{Im} \alpha$, $B := \ker \beta$ and $C := \text{Im} \beta$. Application of the functor $H^i_{yR}$ to the above exact sequence gives the epimorphism $H^i_{yR}(H^{i-1}_{R^+}(K, N)) \to H^i_{yR}(A)$, the monomorphism $\Gamma_{yR}(C) \to \Gamma_{yR}(H^i_{R^+}(K, N))$ and the exact sequence $\Gamma_{yR}(C) \to H^i_{yR}(B) \to H^1_{yR}(H^i_{R^+}(R^t, N)) \to H^1_{yR}(H^{i-1}_{R^+}(K, N)) = H^1_{m_0R}(H^{i-1}_{R^+}(K, N))$ is Artinian and so is $H^1_{yR}(A)$. On the other hand since, by (i), the module $\Gamma_{yR}(H^i_{R^+}(K, N))$ is Artinian, the module $\Gamma_{yR}(C)$ is Artinian. Using [1, Theorem 2.5], the module $H^1_{yR}(H^i_{R^+}(R^t, N))$ is Artinian and then the exact sequence $(*)$ and the previous arguments imply that $H^1_{yR}(B)$ is Artinian. Now, since both $H^1_{yR}(A)$ and $H^1_{yR}(B)$ are Artinian, one can easily deduce that $H^1_{yR}(H^i_{R^+}(M, N)) = H^1_{m_0R}(H^i_{R^+}(M, N))$ is Artinian.

**Proposition 2.10.** Let $\dim R_0 \leq 0$. Then $H^p_{R^+}(M, H^1_{m_0R}(N))$ is Artinian for each
Proof. If \( \dim R_0 = 0 \), then \( H^1_{m_0R}(N) = 0 \) and so the result is clear in this case. Now, assume that \( \dim R_0 = 1 \). By the Grothendieck spectral sequence (see [6, Theorem 11.38]), for each \( p, q \in N_0 \), there is \( E_2^{p,q} := H^p_{R_+}(M, H^q_{m_0R}(N)) \). As \( \dim R_0 = 1 \), we have \( H^p_{m_0R}(N) = 0 \) for all \( q > 1 \) and then \( E_2^{p,q} = 0 \) for all \( q \neq 0, 1 \). Thus we can apply the dual of [9, Ex. 5.2.2] to get the following exact sequence

\[
\cdots \rightarrow j \rightarrow \cdots
\]

and the result has been proved for all values smaller that \( H \) and \( j \).

Using [1, Theorem 2.5], the module \( \Gamma_{R_+} \in i, j = 1, \cdots, n \). We note that by induction hypothesis, the module \( \Gamma_{R_+} \) is Artinian. Now, in view of the above exact sequence, we get our assertion.

**Proposition 2.11.** Let \( \dim R_0 \leq 1 \). Then \( H^j_{m_0R}(M, H^i_{R_+}(N)) \) is Artinian for each \( i, j \in N_0 \).

Proof. If \( \dim R_0 = 0 \), then each finitely generated \( R \)-module is \( m_0 \)-torsion. Thus \( H^i_{R_+}(N) \) is Artinian and so is \( m_0 \)-torsion. Then for each \( j \), there is an isomorphism \( H^j_{m_0R}(M, H^i_{R_+}(N)) \cong \text{Ext}^j_{R_+}(M, H^i_{R_+}(N)) \). One can easily show that the last module is Artinian. Now, assume that \( \dim R_0 = 1 \). We proceed by induction on \( j \). If \( j = 0 \), then we have \( H^0_{m_0R}(M, H^j_{R_+}(N)) = \text{Hom}_R(M, R_+H^j_{R_+}(N)) \). Using [1, Theorem 2.5], the module \( \Gamma_{m_0R}(H^j_{R_+}(N)) \) is Artinian and so one can easily show that \( \text{Hom}_R(M, \Gamma_{m_0R}(H^j_{R_+}(N))) \) is Artinian. Now, we assume that \( j > 0 \) and the result has been proved for all values smaller that \( j \) and we prove it for \( j \).

Since \( M \) is finitely generated, for some positive integer \( t \), there is an exact sequence

\[
0 \rightarrow K \rightarrow R^t \rightarrow M \rightarrow 0
\]

of \( R \)-modules. In view of Lemma 2.2, if we apply the functor \( H^j_{m_0R}(-, H^i_{R_+}(N)) \) to the above exact sequence, we have the following exact sequence

\[
H^{j-1}_{m_0R}(K, H^i_{R_+}(N)) \rightarrow H^j_{m_0R}(M, H^i_{R_+}(N)) \rightarrow H^j_{m_0R}(R^t, H^i_{R_+}(N)).
\]

We note that by induction hypothesis, the module \( H^{j-1}_{m_0R}(K, H^i_{R_+}(N)) \) is Artinian and \( H^j_{m_0R}(R^t, H^i_{R_+}(N)) = 0 \) for all \( j > 1 \) and also \( H^1_{m_0R}(R^t, H^i_{R_+}(N)) \) is Artinian by [1, Theorem 2.5]. Now, in view of the above exact sequence, we get our assertion.

**References**


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