The Characteristics and Descendants of Some Permutation Groups

Nathaniel A. Baylas IV†, Dexter Jane L. Indong∗, and Teofina A. Rapanut⋆

Department of Mathematics and Computer Science
University of the Philippines Baguio
Governor Pack Road, Baguio City 2600, Philippines
†nabaylasiv@upb.edu.ph, ∗djlindong@upb.edu.ph,
⋆tarapanut@upb.edu.ph

Gilbert R. Peralta

Department of Mathematics and Computer Science
University of the Philippines Baguio
Governor Pack Road, Baguio City 2600, Philippines
Current Address: Institut für Mathematik und Wissenschaftliches Rechnen
Karl-Franzens-Universität Graz
Heinrichstrasse 36, 8010 Graz, Austria
grperalta@upb.edu.ph, gilbert.peralta@edu.uni-graz.at

Abstract

Let \( f \) be a complex-valued function defined on a subgroup \( G \) of the symmetric group of a linearly ordered set with \( n \) elements. In this paper, the \( f \)-characteristics and \( f \)-descendants of \( G \) are defined. The \( f \)-characteristic of such group arises from the inner product space of all complex-valued functions on \( G \) and we note that the notion of characteristics generalizes Burnside’s Lemma. Loosely speaking, a characteristic is an average mass of a complex valued function on a permutation group. We give some properties of the characteristics and examples based on the factor groups, convolutions, sign of permutations and roots of unity. The study will then be restricted to functions on the symmetric, alternating and dihedral groups on \( \{1, \ldots, n\} \) whose range lies in the set of nonnegative integers, which are called permutation statistics. Furthermore, we show that the inversion, descent, ascent and major index descendants of the dihedral group coincides.

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1 Introduction and Preliminaries

Let $\Omega$ be a nonempty finite set and $L(\Omega)$ be the $\mathbb{C}$-vector space of all complex-valued functions on $\Omega$. The space $L(\Omega)$ is a pre-Hilbert space endowed with the normalized scalar product $(f, g)_{L(\Omega)} = \frac{1}{|\Omega|} \sum_{x \in \Omega} f(x)\overline{g(x)}$ for $f, g \in L(\Omega)$. For each $f \in L(\Omega)$ we set $\|f\|_{L(\Omega)}^2 = (f, f)_{L(\Omega)}$. The scalar product defined above is used in representation theory for finite groups (see for instance, Hoffman and Humphreys [8] and Ceccherini-Silberstein, et. al. [5]).

Assume that the pair $(X, <)$ is a linearly ordered set, where $X$ is a nonempty finite set with cardinality $|X| = n$, and denote by $S_X$ the set of all bijective functions from $X$ onto itself, that is, $S_X$ is the set of all permutations of $X$. We know that $S_X$ is a group having $n!$ elements called the symmetric group of $X$. The identity permutation in $S_X$ is denoted by $\iota_{S_X}$. We also write the identity permutation by $\iota$ if the set $X$ is clear in the context. If $X = [n] = \{1, 2, \ldots, n\}$, we use the notation $S_n$ for $S_X$. Let $1_G : G \rightarrow \mathbb{C}$ be the constant mapping defined by $1_G(\sigma) = 1$ for all $\sigma \in G$.

Define $\chi(f, G) = \langle f, 1_G \rangle_{L(G)}$ for each $G \leq S_X$ and $f \in L(G)$. The number $\chi(f, G)$ is called the $f$-characteristic of $G$. Note that the inner product in $L(G)$ and the notion of characteristic are related by $\chi(f \cdot \overline{g}, 1_G) = \langle f, g \rangle_{L(G)}$ for each $f, g \in L(G)$.

A function $s \in L(S_X)$ such that $s(\sigma) \in \mathbb{N} \cup \{0\}$ for all $\sigma \in S_X$ is called a permutation statistics, and in this case we have $\chi(s, G) = \|s^2\|_{L(G)}^2$. We can view the number $\chi(s, G)$ as the average mass of the statistics $s$ over the group $G$. In line with this definitions, we remark that the fixed point characteristic of a permutation group is a classical problem in combinatorics. If $\text{fix}(\sigma)$ denotes the number of fixed points of $\sigma$ then Burnside’s lemma implies that the fixed point characteristic of a permutation group $G$ is given by $\chi(\text{fix}, G) = |[n]/\sim|$, where $[n]/\sim$ denotes the set of all equivalence classes, called $G$-orbits, of $[n]$ under the equivalence relation $\sim$ defined as follows: given $i, j \in [n]$, $i \sim j$ if and only if $\sigma(i) = j$ for some $\sigma \in G$ (see Aigner [1]).

The study of permutation statistics was initiated by Euler, on which he considered a classical permutation statistics, the total number of descents of a permutation. Let $x_1 < x_2 < \cdots < x_n$ be a list of elements of $X$ in increasing order. The integer $i \in [n-1]$ is called a descent of $\sigma \in S_X$ if $\sigma(x_i) > \sigma(x_{i+1})$. The set of all descents of a permutation $\sigma$ is called the descent set of $\sigma$ and it is denoted by $\text{Des}(\sigma)$. By definition $\text{Des}(\sigma) \subset [n-1]$. Define $\text{des} : S_X \rightarrow \mathbb{C}$ by $\text{des}(\sigma) = |\text{Des}(\sigma)|$, that is, $\text{des}(\sigma)$ is the total number of descents of $\sigma$. For descent pairs on colored permutation groups we refer the readers to Bagno et.
The inversion set of a permutation $\sigma \in S_X$ is denoted and defined by
\[
\text{Inv}(\sigma) = \bigcup_{j=1}^{n-1} \{i : i < j \text{ and } \sigma(x_i) > \sigma(x_j)\}
\]
Define $\text{inv} : S_X \to \mathbb{C}$ by $\text{inv}(\sigma) = |\text{Inv}(\sigma)|$. The above statistics is called inversion or inversion number. The set $A_X = \{\sigma \in S_X : \text{inv}(\sigma) \in 2\mathbb{Z}\}$ is a subgroup of $S_X$ having index $2$ called the alternating group of $X$. An element of $S_X$ is called an even permutation of $X$ if it is contained in $A_X$, otherwise, it is called an odd permutation of $X$.

The major index $\text{maj} : S_X \to \mathbb{C}$ is the function defined to be the sum of all the descents of $\sigma$, that is, $\text{maj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} i$. Let $M(n, k)$ and $I(n, k)$ be the total number of permutations of $[n]$ having $k$ major index and $k$ inversions, respectively. It can be easily seen that $\text{inv}[S_n] = \text{maj}[S_n] = \{0, 1, \ldots, \binom{n}{2}\}$ for all $n \geq 1$.

**Definition 1.1.** Let $f, g \in L(G)$, where $G$ is a subgroup of $S_X$. Then $f$ and $g$ are said to be equally distributed in $G$ if $f[G] = g[G]$ and
\[
|\{\sigma \in G : f(\sigma) = x\}| = |\{\sigma \in G : g(\sigma) = x\}|
\]
for all $x \in f[G]$. If a statistics $s$ is equally distributed to the inversion statistics in $G$, then $s$ is called Mahonian in $G$. Similarly, if $s$ is equally distributed to the number of descent statistics in $G$, then $s$ is called Eulerian in $G$.

The permutation statistics major index was introduced by Major Percy MacMahon [12]. With the aid of generating functions, MacMahon showed that $M(n, k) = I(n, k)$ for all $k = 0, 1, \ldots, \binom{n}{2}$, that is the major index and inversion number are equally distributed in $S_n$, by proving that
\[
\sum_{\sigma \in S_n} x^{\text{maj}(\sigma)} = \sum_{\sigma \in S_n} x^{\text{inv}(\sigma)}.
\]
Foata [6, 7] gave an alternative proof of the above result by constructing a bijection from the symmetric group with the property that the major index of a permutation equals the inversion number of its image. Recently, Assaf [2] introduced a statistic called the $k$-major index which generalizes the inversion number and major index, extended this notion to standard Young tableaux for $k \leq 3$ and cited some connections to Macdonald polynomials. By constructing a family of bijections having a recursive structure similar to Foata’s bijection, he showed that the $k$-major index is Mahonian statistics. Other examples of permutation statistics are presented in Bóna [4].

For inversion characteristic of the symmetric and alternating groups of linearly ordered sets we have the following result.
The analogous results regarding alternating groups can be handled in a similar way.

Given \( G \leq S_X \) and \( f \in L(G) \) we define the set \( \Delta(f, G) = \{ H : H \leq G \) and \( \chi(f, H) = \chi(f, G) \} \) and its cardinality is denoted by \( \nu(f, G) \). An element of \( \Delta(f, G) \) is called an \( f\)-descendant of \( G \). Loosely speaking, a subgroup \( H \) of \( G \) is an \( f\)-descendant of \( G \) if the average mass of \( f \) over \( H \) and \( G \) are the same.

Let \( D_n = \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle \), where

\[
a = \begin{pmatrix}
1 & 2 & \cdots & n-1 & n \\
2 & 3 & \cdots & n & 1
\end{pmatrix}, \quad b = \begin{pmatrix}
1 & 2 & 3 & \cdots & n-1 & n \\
1 & n & n-1 & \cdots & 3 & 2
\end{pmatrix},
\]

be the dihedral group of degree \( n \), that is, the group of isometries of a regular \( n \)-gon (see Hungerford [9]). We note that the dihedral group can be written as a semi-direct product of the cyclic groups generated by \( a \) and \( b \), that is, \( D_n = \langle a \rangle \ltimes \langle b \rangle \) (see Kurzweil [11]). It was shown in [13] that the total number of inversion descendants of the dihedral group \( D_n \) is equal to the number of positive divisors of \( n \), i.e., \( \nu(\text{inv}, D_n) = \tau(n) \) for \( n \geq 3 \).

The paper is organized as follows. In Section 2, we give some examples based on convolutions, factor groups, sign of permutations and the roots of unity. The rest of the paper will then be devoted to the study of characteristics of the permutation groups \( S_n, A_n \), and \( D_n \) under the permutation statistic descent, ascent, major index and together with their convolutions. Also, the descendants of \( D_n \) with respect to these permutation statistic will be considered. Interestingly, they are precisely the inversion descendants of the dihedral group and satisfy the property that the reverse permutation of an element in a descendant lies also in the said descendant.
2 Examples on Convolutions, Factor Groups, Signed Characteristics and Roots of Unity

For separable functions on a product of permutation groups, we have the following theorem.

**Theorem 2.1.** Let $f \in L(G)$ and $g \in L(H)$ where $G$ and $H$ are subgroups of $S_X$. Define $f \otimes g : G \times H \to \mathbb{C}$ by $(f \otimes g)(\sigma, \tau) = f(\sigma)g(\tau)$. Then $\chi(f \otimes g, G \times H) = \chi(f, G)\chi(g, H)$. In particular, $\chi(f \otimes g, G \times H) = \chi(g \otimes f, H \times G)$.

**Proof.** For such pairs $f$ and $g$ we have

$$\chi(f \otimes g, G \times H) = \frac{1}{|G \times H|} \sum_{(\sigma, \tau) \in G \times H} (f \otimes g)(\sigma, \tau) = \left( \frac{1}{|G|} \sum_{\sigma \in G} f(\sigma) \right) \left( \frac{1}{|H|} \sum_{\tau \in H} g(\tau) \right) = \chi(f, G)\chi(g, H).$$

This completes the proof. \qed

**Definition 2.2.** Let $G$ be a subgroup of $S_X$ and $f, g \in L(G)$. The normalized convolution $f \ast g : G \to \mathbb{C}$ of $f$ and $g$ is defined by

$$(f \ast g)(\sigma) = \frac{1}{|G|} \sum_{\tau \in G} f(\sigma\tau^{-1})g(\tau).$$

We note $L(G)$ is an algebra over $\mathbb{C}$ with unity (see [5] for some properties regarding convolutions). In general, the convolution is not commutative. The succeeding theorem states that the characteristic of a permutation group $G$ with respect to a convolution does not depend on the order of the functions on which we convolute.

**Theorem 2.3.** If $G \leq S_X$ and $f, g \in L(G)$ then

$$\chi(f \ast g, G) = \chi(f, G)\chi(g, G) = \chi(g \ast f, G).$$

**Proof.** Since $G\tau^{-1} = G$ for all $\tau \in G$ it follows that

$$\chi(f \ast g, G) = \frac{1}{|G|} \sum_{\sigma \in G} (f \ast g)(\sigma) = \frac{1}{|G|^2} \sum_{\sigma \in G} \sum_{\tau \in G} f(\sigma\tau^{-1})g(\tau) = \frac{1}{|G|^2} \sum_{\tau \in G} \sum_{\sigma \in G} g(\tau)f(\sigma\tau^{-1}) = \frac{1}{|G|} \sum_{\tau \in G} g(\tau)\chi(f, G) = \chi(g, G)\chi(f, G).$$

By symmetry, we also have $\chi(g, G)\chi(f, G) = \chi(f, G)\chi(g, G) = \chi(g \ast f, G)$ and this proves the theorem. \qed
Now let us consider factor groups. Let $H \trianglelefteq G \leq S_X$, $G/H = \{H\sigma : \sigma \in G\}$ and $\sigma_1, \ldots, \sigma_t$ be a complete list of representatives for the right cosets of $H$ in $G$, so that $G$ is the disjoint union $H\sigma_1 \cup H\sigma_2 \cup \cdots \cup H\sigma_t$. Let $\gamma : G \to G/H$ be the mapping defined by $\gamma(\sigma) = H\sigma$. Given $\tilde{f} \in L(G/H)$ define $f \in L(G)$ by $f = \tilde{f} \circ \gamma$. Thus, $f(\sigma) = f(H\sigma_i)$ whenever $\sigma \in H\sigma_i$ for precisely one $i = 1, \ldots, t$. With this definition, we have

$$\sum_{\sigma \in H\sigma_i} f(\sigma) = \sum_{\sigma \in H\sigma_i} \tilde{f}(H\sigma_i) = |H|\tilde{f}(H\sigma_i),$$

for all $i = 1, \ldots, t$. Thus

$$\chi(\tilde{f}, G/H) = \frac{1}{|G/H|} \sum_{i=1}^t \tilde{f}(H\sigma_i) = \frac{1}{|G|/|H|} \sum_{i=1}^t \left( \frac{1}{|H|} \sum_{\sigma \in H\sigma_i} f(\sigma) \right)$$

$$= \frac{1}{|G|} \sum_{i=1}^t \sum_{\sigma \in H\sigma_i} f(\sigma) = \chi(f, G).$$

On the other hand, given $f \in L(G)$ let

$$H = \{\sigma \in G : f(\sigma \tau) = f(\tau) \text{ for all } \tau \in G\}. \quad (1)$$

We claim that $H \leq G$. Indeed, it is clear that $e \in H$ and if $\sigma_1, \sigma_2 \in H$ we have $f((\sigma_1\sigma_2)\tau) = f(\sigma_1(\sigma_2\tau)) = f(\sigma_2\tau) = f(\tau)$ for all $\tau \in G$ and so $\sigma_1\sigma_2 \in H$. If $\sigma \in H$ we have $f(\sigma^{-1}\tau) = f(\sigma^{-2}\tau) = f(\sigma^{-2}\tau)$. By induction, and since $|X| = n$ it follows that $f(\sigma^{-1}\tau) = f((\sigma^{-1})^{[n]}\tau) = f(\tau)$ for all $\tau \in G$ and so $\sigma^{-1} \in H$.

Suppose that $H$ is a normal subgroup of $G$. Define $\tilde{f} \in L(G/H)$ by $\tilde{f}(H\sigma) = f(\sigma)$. Note that $\tilde{f}$ is well-defined since the image of a right coset of $H$ under $\tilde{f}$ does not depend on the representative of the coset, that is, $\tilde{f}(H(\eta\sigma)) = f(\eta\sigma) = f(\sigma)$ for all $\eta \in H$. Denote by $\sigma_1, \ldots, \sigma_t$ a complete set of representatives of the right cosets of $H$. Moreover, we have $\tilde{f} \circ \gamma = f$. If $\sigma \in H\sigma_i$ then $H\sigma = H\sigma_i$ and so $f(\sigma) = \tilde{f}(H\sigma) = \tilde{f}(H\sigma_i)$. This implies that

$$\sum_{\sigma \in H\sigma_i} f(\sigma) = |H|\tilde{f}(H\sigma_i)$$

Thus, we have proved the following theorem.

**Theorem 2.4.** Let $H \trianglelefteq G \leq S_X$, $\tilde{f} \in L(G/H)$ and $\gamma : G \to G/H$ be the homomorphism given by $\gamma(\sigma) = H\sigma$. If

$$f = \tilde{f} \circ \gamma$$

(2)
then
\[ \chi(f, G) = \chi(\tilde{f}, G/H). \] (3)

Also, given \( f \in L(G) \), if the subgroup \( H \) of \( G \) given by (1) is normal, then there is a function \( f \in L(G/H) \) such that (2) and (3) are satisfied.

The sign \( \text{sgn}(\sigma) \) of a permutation \( \sigma \in S_X \) is defined to be \( \text{sgn}(\sigma) = 1 \) if \( \sigma \in A_X \) and \( \text{sgn}(\sigma) = -1 \) otherwise. The characteristic \( \chi(\text{sgn} \cdot f, G) \) is called the signed \( f \)-characteristic of \( G \). The following theorem states that if the \( f \)-characteristics of \( S_X \) and \( A_X \) are equal then the signed \( f \)-characteristic of \( S_X \) must be zero.

**Theorem 2.5.** If \( A_X \) is an \( f \)-descendant of \( S_X \) then \( \chi(\text{sgn} \cdot f, S_X) = 0 \).

**Proof.** Note that
\[
\chi(\text{sgn} \cdot f, S_X) = \frac{1}{|S_X|} \sum_{\sigma \in A_X} f(\sigma) - \frac{1}{|S_X|} \sum_{\sigma \in S_X \setminus A_X} f(\sigma)
= \frac{2}{|S_X|} \sum_{\sigma \in A_X} f(\sigma) - \frac{1}{|S_X|} \sum_{\sigma \in S_X} f(\sigma)
\]
and so \( \chi(\text{sgn} \cdot f, S_X) = \chi(f, A_X) - \chi(f, S_X) = 0. \) \( \Box \)

**Remark 2.6.** From Theorem 1.2 we know that \( \chi(\text{inv}, S_X) = \chi(\text{inv}, A_X) \) for all \( n \geq 4 \) and it now follows from the above theorem that \( \chi(\text{sgn} \cdot \text{inv}, S_X) = 0 \).

Let \( 1, \omega, \ldots, \omega^{n-1} \), where \( \omega = e^{2\pi i/n} = \cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n}) \), be the \( n \)th-roots of unity and \( T \) be a nonempty subset of \([n]\). Define \( g_T : S_n \to \mathbb{C} \) by
\[
g_T(\sigma) = \sum_{i \in T} \omega^{\sigma(i)}. \]

**Theorem 2.7.** For \( n \geq 2 \) we have
\[
\chi(g_T, S_n) = \frac{|T|}{n} \cdot \frac{\omega^n - 1}{\omega - 1}. \] (4)

If \( n \geq 4 \) the \( \chi(g_T, S_n) = \chi(g_T, A_n) \) and if \( n \geq 3 \) we have \( \chi(g_T, D_n) = \chi(g_T, S_n) \). In particular, \( A_n \) and \( D_n \) are \( g_T \)-descendants of \( S_n \) for \( n \geq 4 \) and for \( n \geq 3 \), respectively.

**Proof.** For a fix \( i \in T \), let \( C_j = \{ \sigma \in S_n : \sigma(i) = j \} \) for each \( j = 1, \ldots, n \). Then \( S_n \) is the disjoint union \( C_1 \cup C_1 \cup \cdots \cup C_n \) and since \( |C_j| = (n - 1)! \) for each \( j \) we obtain
\[
\chi(g_T, S_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{i \in T} \omega^{\sigma(i)} = \frac{1}{n!} \sum_{i \in T} \sum_{j=1}^n \sum_{\sigma \in C_j} \omega^{\sigma(i)} = \frac{1}{n!} \sum_{i \in T} \frac{\omega^n - 1}{\omega - 1}.
\]
and (4) now follows. The result regarding the alternating group can be shown in a similar way.

The last conclusion follows from \( \{a^k(i) : 0 \leq k \leq n - 1\} = [n] \) and \( \{a^kb(i) : 0 \leq k \leq n - 1\} = [n] \) for each \( i \in T \). Indeed, if \( i \in [n] \) then

\[
a^k(i) = \begin{cases} 
  i + k, & \text{if } 0 \leq k \leq n - i, \\
  i + k - n, & \text{if } n - i + 1 \leq k \leq n - 1.
\end{cases}
\]

Similarly, since \( b(i) \in [n] \) it follows from what we have just prove that \( \{a^k(b(i)) : 0 \leq k \leq n - 1\} = [n] \)

As a consequence, the signed \( g_T \)-characteristic of \( S_n \) is zero for all \( n \geq 4 \), that is, \( \chi(\text{sgn} \cdot g_T, S_n) = 0 \) whenever \( n \geq 4 \).

3 Descents

It can be easily seen, using a similar argument as in the proof of Theorem 1.2, that for \( n \geq 1 \) we have \( \chi(\text{des}, S_n) = \chi(\text{des}, S_X) \) and \( \chi(\text{des}, A_n) = \chi(\text{des}, A_X) \) for any linearly ordered set \( X \) with cardinality \( n \).

For each \( j = 1, 2, \ldots, n \), let \( O_j = \{ \sigma \in S_n : \sigma(1) = j \} \) and \( X_j = [n] \setminus \{j\} \). Then \( O_j \cap O_i = \emptyset \) for \( i \neq j \) and \( S_n = \bigcup_{j=1}^n O_j \). If \( \sigma \in O_j \), let \( \tau_\sigma \in S_{X_j} \) be defined as \( \tau_\sigma(x_i) = \sigma(i+1) \), where \( \{x_1, \ldots, x_{n-1}\} \) is a list in increasing order of \( X_j \). The above partition and one-to-one correspondence between the elements of \( O_j \) and \( S_{X_j} \) were considered in [10]. Also, we have \( \text{inv}(\sigma) = j - 1 + \text{inv}(\tau_\sigma) \) whenever \( \sigma \in O_j \).

To determine the descent characteristic of \( S_n, A_n \) and \( D_n \) for large values of \( n \), we will further subdivide each \( O_j \) into \( n - 1 \) subsets. Given \( j \), for \( i \neq j \) we let \( O_{ji} = \{ \sigma \in O_j : \sigma(2) = i \} \). One can easily see that \( O_{ji} \cap O_{jk} = \emptyset \) for \( i \neq k \) and \( O_j = \bigcup_{j \neq j} O_{ji} \). Furthermore, given \( \sigma \in O_{ji} \) we have \( \text{des}(\sigma) = \delta_{ji} + \text{des}(\tau_\sigma) \), where \( \delta_{ji} = 1 \) if \( i < j \) and \( \delta_{ji} = 0 \) if \( i > j \). Since \( \chi(\text{des}, S_{n-1}) = \chi(\text{des}, S_{X_j}) \) we have \( \sum_{\sigma \in O_{ji}} \text{des}(\tau_\sigma) = (n - 1)! \chi(\text{des}, S_{n-1}) \).

Before giving a compact formula for the descent characteristic of the symmetric groups, we state the following lemma which presents a recurrence formula for this characteristic.

**Lemma 3.1.** The descent characteristic of the symmetric groups satisfy the following recurrence relation \( \chi(\text{des}, S_1) = 0 \), \( \chi(\text{des}, S_n) = \frac{1}{2} + \chi(\text{des}, S_{n-1}) \), for \( n \geq 2 \).

**Proof.** Let \( n \geq 2 \). From the above discussion, the total mass of the descent in \( O_j \) is given as

\[
\sum_{\sigma \in O_j} \text{des}(\sigma) = \sum_{i \neq j} \sum_{\sigma \in O_{ji}} \text{des}(\sigma) = \sum_{i \neq j} \sum_{\sigma \in O_{ji}} [\delta_{ji} + \text{des}(\tau_\sigma)] = (j - 1)(n - 2)! + (n - 1)! \chi(\text{des}, S_{n-1}).
\]
Getting the sum of all the total masses of the descent in \( O_1, \ldots, O_n \) and then averaging we obtain \( \chi(\text{des}, S_n) = \frac{1}{2} + \chi(\text{des}, S_{n-1}) \).

**Theorem 3.2.** The descent characteristic of the symmetric group \( S_n \) is given by \( \chi(\text{des}, S_n) = \frac{n-1}{2} \) for all \( n \geq 1 \).

**Proof.** The proof of this theorem follows from the previous lemma and by induction. \( \square \)

**Example 3.3.** If \( n \geq 1 \) then

\[
\sum_{k=1}^{n} \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n (k-1) = \binom{n}{2} (n-1)!. \tag{5}
\]

To prove this, for each \( k = 1, \ldots, n \) let \( A(n, k) \) denotes the number of permutations in \( S_n \) having \( k-1 \) descents. The number \( A(n, k) \) is called an Eulerian number. An explicit formula for the Eulerian numbers is given by [4]

\[
A(n, k) = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n.
\]

Since \( n! \chi(\text{des}, S_n) = \sum_{k=1}^{n} A(n, k) (k-1) \), (5) follows from Theorem 3.2.

From Theorem 2.3, the characteristic of \( S_n \) with respect to the convolution of the permutation statistics \( \text{inv} \) and \( \text{des} \) is given in the succeeding corollary.

**Corollary 3.4.** For every \( n \geq 1 \), \( \chi(\text{inv} * \text{des}, S_n) = \frac{n(n-1)^2}{8} \).

Now, we exploit the partitioning method presented in the above discussions to find the descent characteristic of the alternating group \( A_n \). The following theorem shows that for \( n \geq 4 \), \( A_n \) is a descent descendant of \( S_n \).

**Theorem 3.5.** For each \( n \geq 4 \), we have

\[
\chi(\text{des}, A_n) = \chi(\text{des}, S_n). \tag{6}
\]

In particular, \( \chi(\text{inv} * \text{des}, A_n) = \chi(\text{inv} * \text{des}, S_n) \) for all \( n \geq 4 \).

**Proof.** One can easily verify that the formula (6) holds for \( n = 4 \). We assume that the formula holds for \( n-1 \). We note that for each \( O_{ji} \), half of its elements are even and half are odd. To prove this, consider a fixed \( O_{ji} \) and let

\[
k = |\{ j : 1 < j \text{ and } \sigma(1) > \sigma(j) \}| + |\{ i : 2 < i \text{ and } \sigma(2) > \sigma(i) \}|.
\]

Let \( X_{ji} = [n] \setminus \{i, j\} \). Given \( \sigma \in O_{ji} \) let \( \varsigma_\sigma \in S_{X_{ji}} \) be defined by \( \varsigma_\sigma(b_i) = \sigma(i+2) \), where \( \{b_1, \ldots, b_{n-2}\} \) is a list of elements of \( X_{ji} \) in increasing order. Thus,
inv(σ) = k + inv(ςσ). The claim follows from this equation since k is fixed and half of the elements of S_{X_j} are even and half are odd (here the assumption n ≥ 4 is needed).

From the equality inv(σ) = j − 1 + inv(τσ) we can show that \{τσ : σ ∈ O_j \cap A_n\} = A_{X_j} if j is odd and \{τσ : σ ∈ O_j \cap A_n\} = S_{X_j} − A_{X_j} if j is even. Hence

\[
\sum_{\sigma \in O_j \cap A_n} des(\tau_\sigma) = \sum_{\sigma \in A_{X_j}} des(\sigma) = \frac{(n-1)!}{2} \chi(des, A_{n-1})
\]

if j is odd and from the induction hypothesis we have

\[
\sum_{\sigma \in O_j \cap A_n} des(\tau_\sigma) = \sum_{\sigma \in S_{X_j}} des(\sigma) - \sum_{\sigma \in A_{X_j}} des(\sigma) = (n-1)! \chi(des, S_{n-1}) - \frac{(n-1)!}{2} \chi(des, A_{n-1})
\]

\[
= \frac{(n-1)!}{2} \chi(des, A_{n-1})
\]

if j is even.

Since des(σ) = δ_{ji} + des(τσ) it follows that

\[
\sum_{\sigma \in O_j \cap A_n} des(\sigma) = \sum_{i \neq j} \sum_{\sigma \in O_{j_i} \cap A_n} des(\sigma) = \sum_{i \neq j} \sum_{\sigma \in O_{j_i} \cap A_n} [δ_{ji} + des(τ_\sigma)]
\]

\[
= \frac{(j-1)(n-2)!}{2} + \frac{(n-1)!}{2} \chi(des, A_{n-1}).
\]

Thus, taking the total mass of des in A_n = \bigcup_{j=1}^{n} (O_j \cap A_n)

\[
\sum_{\sigma \in A_n} des(\sigma) = \sum_{j=1}^{n} \sum_{\sigma \in O_j \cap A_n} des(\sigma) = \frac{n!}{2} \left( \frac{1}{2} + \chi(des, S_{n-1}) \right).
\]

Averaging gives us the desired result \chi(des, A_n) = \chi(des, S_n). This completes the proof of the induction step, and in turn the proof of the theorem. The second part is straightforward from Theorem 2.3.

\[\square\]

**Corollary 3.6.** For all \( n \geq 4 \), \( \chi(sgn \cdot des, S_n) = 0 \).

We define the permutation \( \beta \in S_X \) by \( \beta(x_i) = x_{n-i+1} \). The mapping \( B : S_X \to S_X \) defined by \( B\sigma = \sigma\beta \) is bijective. The image \( B\sigma \) of \( \sigma \) under \( B \) is called the backward or reverse permutation of \( \sigma \).

**Definition 3.7.** A subgroup \( G \leq S_X \) is said to be a \( B \)-group if \( G \) is closed under \( B \), that is, \( B\sigma \in G \) for all \( \sigma \in G \).
One can verify that $S_n$ is a $B$-group for $n \geq 1$, $D_n$ is a $B$-group for $n \geq 3$, and $A_n$ is a $B$-group if and only if $n \equiv 0, 1 \pmod{4}$. The following theorem characterizes the set of descents of a backward permutation.

**Theorem 3.8.** For each $\sigma \in S_n$ we have $\text{Des}(B\sigma) = \{n - i : i \notin \text{Des}(\sigma)\}$ and $\text{des}(B\sigma) + \text{des}(\sigma) = n - 1$.

*Proof.* Note that $B\sigma(i) = \sigma(n - i + 1)$. Thus $B\sigma(n - i) = \sigma(i + 1)$ and $B\sigma(n - i + 1) = \sigma(i)$. Hence $n - i \in \text{Des}(B\sigma)$ if and only if $\sigma(i + 1) > \sigma(i)$, and the latter inequality is true if and only if $i \notin \text{Des}(\sigma)$. Also, $\text{des}(B\sigma) = |\text{Des}(B\sigma)| = n - 1 - |\text{Des}(\sigma)| = n - 1 - \text{des}(\sigma)$. \hfill $\square$

For $B$-groups, their descent characteristics are presented in the following theorem.

**Theorem 3.9.** If $G$ is a $B$-subgroup of $S_n$ then $\chi(\text{des}, G) = \frac{1}{2}(n - 1)$ and so $G$ is a descent descendant of $S_n$. In particular, $\chi(\text{des}, S_n) = \frac{n - 1}{2}$ for all $n \geq 1$ and $D_n$ is a descent descendant of $S_n$ for all $n \geq 3$.

*Proof.* There exists two disjoint subsets $G_1$ and $G_2$ of $G$ such that $G = G_1 \cup G_2$, $|G_1| = |G_2|$, and $B[G_1] = G_2$. Hence

$$
\chi(\text{des}, G) = \frac{1}{|G|} \sum_{\sigma \in G_1} [\text{des}(\sigma) + \text{des}(B\sigma)] = \frac{(n - 1)|G_1|}{|G|} = \frac{n - 1}{2},
$$

and this establishes the theorem. \hfill $\square$

We note that the converse of the above theorem does not hold in general. For example, consider the alternating group $A_6$. By Theorem 3.5, $\chi(A_6) = \frac{5}{2}$. But note that $A_6$ is not a $B$-group since $\text{inv}(\beta) = \binom{6}{2} = 15$ and so $B\beta = \beta \notin A_6$.

Now, let us find the value of $\nu(\text{des}, D_n)$. The subgroups of the dihedral group $D_n$ are either of the following:

(a) cyclic subgroup of the form $\langle a^j \rangle$ or $\langle a^j b \rangle$, for some $j = 0, 1, \ldots, n - 1$.

(b) dihedral subgroup of the form $D_m^j := \langle a^{n/m} \rangle \rtimes \langle a^j b \rangle$ for some $j = 0, 1, \ldots, n/m - 1$ where $m|n$.

In [13], it was shown that $a^j b = B a^{j + 1}$ for $j = 0, 1, \ldots, n - 1$. Since $\text{Des}(\iota) = \emptyset$ and $\text{Des}(a^j) = \{n - j\}$ for $j = 1, 2, \ldots, n - 1$ it follows that

$$
\text{des}(a^j b) = n - 1 - \text{des}(a^{j + 1}) = \begin{cases} n - 1, & \text{if } j = n - 1; \\ n - 2, & \text{if } j \neq n - 1. \end{cases}
$$

Further, we have $\Delta(\text{inv}, D_n) \subset \Delta(\text{des}, D_n)$, where

$$
\Delta(\text{inv}, D_n) = \{\langle a^{n-1} b \rangle\} \cup \{D_m^{n/m-1} : m|n \text{ and } m \geq 2\}
$$

and so we have $\tau(n) \leq \nu(\text{des}, D_n)$. A stronger result is given in the following theorem.
Theorem 3.10. For all $n \geq 3$, we have $\Delta(\text{des}, D_n) = \Delta(\text{inv}, D_n)$ and so $\nu(\text{des}, D_n) = \tau(n)$. Furthermore, $H \in \Delta(\text{des}, D_n)$ if and only if $H$ is a $B$-subgroup of $D_n$.

Proof. For cyclic subgroups of the form $\langle a^j \rangle$, where $j|n$ and $n = jm$, we have

$$\chi(\text{des}, \langle a^j \rangle) = \frac{1}{m} \sum_{i=1}^{m} \text{des}(a^{ji}) = \frac{m-1}{m} < \chi(\text{des}, D_n).$$

On the other hand, for cyclic groups generated by mirror reflections, we have

$$\chi(\text{des}, \langle a^j b \rangle) = \frac{1}{2} \text{des}(a^j b) = \begin{cases} \chi(\text{des}, D_n), & \text{if } j = n-1; \\ \chi(\text{des}, D_n) - \frac{1}{2}, & \text{if } j \neq n-1. \end{cases}$$

For dihedral subgroups of $D_n$, we consider the following two cases. First, let us assume that $j = n/m - 1$. Then from the previous remark, $\chi(\text{des}, D_m^{n/m-1}) = \chi(\text{des}, D_n)$. If $j \neq n/m - 1$, we have

$$\chi(\text{des}, D_m) = \frac{1}{2m} \left[ \sum_{i=0}^{m-1} \text{des}(a^{ni/m}) + \sum_{i=0}^{m-1} \text{des}(a^{i+ni/m}b) \right]$$

$$= \frac{1}{2m} \left[ (m-1) + \sum_{i=0}^{m-1} (n-2) \right] = \chi(\text{des}, D_n) - \frac{1}{2m}.$$

Therefore $\{\langle a^{n-1} b \rangle \} \cup \{D_m^{n/m-1} : m|n \text{ and } m \geq 2 \} = \Delta(\text{des}, D_n)$. Hence $\nu(\text{des}, D_n) = \tau(n)$. 

4 Ascents

Let $\sigma \in S_X$. An element $i \in [n-1]$ is called an ascent of $\sigma$ if $\sigma(x_i) < \sigma(x_{i+1})$. The total number of ascents of $\sigma$ is denoted by $\text{asc}(\sigma)$. It can be easily seen that $\text{asc}(\sigma) + \text{des}(\sigma) = n - 1$. For, $B$-groups we have the following result.

Theorem 4.1. If $G$ is a $B$-subgroup of $S_n$ then $\chi(\text{des}, G) = \chi(\text{asc}, G)$. In particular, $\chi(\text{asc} \ast \text{des}, G) = [\chi(\text{des}, G)]^2$ and $\chi(\text{inv} \ast \text{des}, G) = \chi(\text{inv} \ast \text{asc}, G)$.

Proof. We have

$$\chi(\text{asc}, G) = \frac{1}{|G|} \sum_{\sigma \in G} [(n-1) - \text{des}(\sigma)] = (n-1) - \frac{n-1}{2} = \chi(\text{des}, G),$$

which proves the theorem. 

Using Theorem 2.5, Theorem 3.5, and the equation $\text{asc}(\sigma) = n-1 - \text{des}(\sigma)$ we obtain the following theorem.
Theorem 4.2. For each $n \geq 4$ we have $\chi(\text{asc}, A_n) = \chi(\text{asc}, S_n)$. In particular, $A_n$ is an ascent descendant of $S_n$ and $\chi(\text{sgn} \cdot \text{asc}, S_n) = 0$ for all $n \geq 4$.

If $H \in \Delta(\text{des}, D_n)$ then $H$ is a $B$-group and $\chi(\text{asc}, H) = \chi(\text{des}, H) = \chi(\text{des}, D_n) = \chi(\text{asc}, D_n)$. Hence $H \in \Delta(\text{asc}, D_n)$ so that $\Delta(\text{des}, D_n) \subset \Delta(\text{asc}, D_n)$.

On the other hand, if $H \in \Delta(\text{asc}, D_n)$ it follows that $\chi(\text{asc}, H) = \chi(\text{asc}, D_n) = \chi(\text{des}, D_n)$. Thus

$$\chi(\text{des}, H) = \frac{1}{|H|} \sum_{\sigma \in H} \text{des}(\sigma) = \frac{1}{|H|} \sum_{\sigma \in H} [n - 1 - \text{asc}(\sigma)]$$

$$= n - 1 - \chi(\text{asc}, H) = \chi(\text{des}, D_n),$$

and so $H \in \Delta(\text{des}, D_n)$. Thus we have proved the following theorem.

Theorem 4.3. For each $n \geq 3$, we have $\Delta(\text{asc}, D_n) = \Delta(\text{des}, D_n)$.

5 Major Indices

First, we have the following general result regarding the characteristics of a finite group $G$ under two equally distributed complex-valued functions.

Theorem 5.1. Let $G \leq S_X$ and $f, g \in L(G)$. If $f$ and $g$ are equally distributed in $G$ then $\chi(f, G) = \chi(g, G)$ and $\chi(f \ast g, G) = \chi(f, G)^2$.

Proof. For each $x \in f[G]$ define $F_x = \{\sigma \in G : f(\sigma) = x\}$ and $G_x = \{\sigma \in G : g(\sigma) = x\}$. Thus

$$\chi(f, G) = \frac{1}{|G|} \sum_{\sigma \in G} f(\sigma) = \frac{1}{|G|} \sum_{x \in f[G]} \sum_{\sigma \in F_x} f(\sigma)$$

$$= \frac{1}{|G|} \sum_{x \in f[G]} x|F_x| = \frac{1}{|G|} \sum_{x \in g[G]} x|G_x| = \chi(g, G),$$

since $|F_x| = |G_x|$ and $f[G] = g[G]$. □

Corollary 5.2. If $s_m$ and $s_e$ are Mahonian and Eulerian statistics on $G$, respectively, then $\chi(s_m, G) = \frac{n(n-1)}{4}$ and $\chi(s_e, G) = \frac{n-1}{2}$. Moreover, for each $f \in L(G)$ we have $\chi(f \ast s_m, G) = \frac{n(n-1)}{4}\chi(f, G)$ and $\chi(f \ast s_e, G) = \frac{n-1}{2}\chi(f, G)$.

Using the previous corollary we have the following result.

Theorem 5.3. For each $n \geq 1$, $\chi(\text{maj}, S_n) = \frac{1}{2}(n)$ and $\chi(\text{inv} \ast \text{maj}, S_n) = \chi(\text{inv}, S_n)^2 = \frac{n^2(n-1)^2}{16}$.

The following theorem is concerned with alternating groups.
Theorem 5.4. For each $n \geq 4$, $\chi(\text{maj}, A_n) = \frac{1}{2} \binom{n}{2}$.

Proof. Given $\sigma \in O_j \cap A_n$, we have $\text{maj}(\sigma) = \text{des}(\sigma) + \text{maj}(\tau_{\sigma})$ since $i + 1 \in \text{Des}(\sigma)$ if and only if $i \in \text{Des}(\tau_{\sigma})$. In a similar way, as in the proof of Theorem 3.5 we have

$$\sum_{\sigma \in O_j \cap A_n} \text{maj}(\tau_{\sigma}) = \binom{n-1}{2} \chi(\text{maj}, A_{n-1})$$

and

$$\sum_{\sigma \in O_j \cap A_n} \text{des}(\tau_{\sigma}) = \frac{(n-1)!(n-2)}{4}.$$  

Using equations (7) and (8) we obtain

$$\sum_{\sigma \in O_j \cap A_n} \text{maj}(\sigma) = \sum_{i<j} \sum_{\sigma \in O_j \cap A_n} \left[ \delta_{ji} + \text{des}(\tau_{\sigma}) + \text{maj}(\tau_{\sigma}) \right]$$

$$= \frac{(j-1)(n-2)!}{2} + \frac{(n-1)!}{2} \chi(\text{maj}, A_{n-1}) + \frac{(n-1)!(n-2)}{4}.$$  

From the induction hypothesis, $\chi(\text{maj}, A_n) = \frac{2n-1}{2} + \chi(\text{maj}, A_{n-1}) = \frac{1}{2} \binom{n}{2}$.  

Next, we are going to relate the major index of a permutation with the major index of its backward permutation. By Theorem 3.8 we have

$$\text{maj}(B\sigma) = \sum_{i \notin \text{Des}(\sigma)} (n-i) = \sum_{i \notin \text{Des}(\sigma)} n - \binom{n}{2} - \sum_{i \in \text{Des}(\sigma)} i$$

$$= \binom{n}{2} - n \text{des}(\sigma) + \text{maj}(\sigma).$$

Theorem 5.5. For each $n \geq 3$, we have $\chi(\text{maj}, D_n) = \frac{1}{2} \binom{n}{2}$.

Proof. Since $D_n$ is a $B$-group we have

$$\sum_{\sigma \in D_n} \text{maj}(\sigma) = \sum_{k=0}^{n-1} [\text{maj}(a^k) + \text{maj}(B a^k)]$$

$$= \binom{n}{2} + \sum_{k=1}^{n-1} \left[ \binom{n}{2} - n \text{des}(a^k) + 2 \text{maj}(a^k) \right]$$

$$= n \binom{n}{2} - n(n-1) + 2 \sum_{k=1}^{n-1} (n-k) = n \binom{n}{2}.$$  

The theorem follows by dividing the order of the dihedral group.  

\[\square\]
For \( j = 0, 1, \ldots, n - 2 \) it follows that \( \text{maj}(a^j b) = \text{maj}(Ba^{j+1}) = \binom{n}{2} - j - 1. \) Also, \( \text{maj}(a^{n-1} b) = \binom{n}{2}. \) This will be used in counting the number of major index descendants of the dihedral group \( D_n. \)

**Theorem 5.6.** For all \( n \geq 3, \) we have \( \Delta(\text{inv}, D_n) = \Delta(\text{maj}, D_n) \) and so \( \nu(\text{maj}, D_n) = \tau(n). \) Furthermore, \( H \in \Delta(\text{maj}, D_n) \) if and only if \( H \) is a \( B \)-subgroup of \( D_n. \)

**Proof.** For cyclic groups generated by a rotation, we have

\[
\chi(\text{maj}, \langle a^j \rangle) = \frac{1}{m} \sum_{i=1}^{m} \text{maj}(a^{ji}) = \frac{1}{m} \sum_{i=1}^{m} (n - ji) = \frac{n - j}{2} < \chi(\text{maj}, D_n),
\]

where \( jm = n. \) The major index characteristic of cyclic subgroups generated by mirror reflection is

\[
\chi(\text{maj}, \langle a^j b \rangle) = \begin{cases} 
\chi(\text{maj}, D_n) - \frac{j+1}{2}, & \text{if } j \neq n - 1; \\
\chi(\text{maj}, D_n), & \text{if } j = n - 1; 
\end{cases}
\]

Next, we will compute the major index characteristics of dihedral subgroups of \( D_n. \) We have

\[
\chi(\text{maj}, D_m^{n/m-1}) = \frac{1}{2m} \left[ \sum_{i=1}^{m} \text{maj}(a^{ni/m}) + \sum_{i=0}^{m-2} \text{maj}(a^{n(i+1)/m-1} b) + \binom{n}{2} \right]
\]

\[
= \chi(\text{maj}, D_m).
\]

Similarly, for each \( j = 0, 1, \ldots, \frac{n}{m} - 2 \) we have

\[
\chi(\text{maj}, D_m^j) = \frac{1}{2m} \left[ \sum_{i=1}^{m} \text{maj}(a^{ni/m}) + \sum_{i=0}^{m-1} \text{maj}(a^{i+ni/m} b) \right]
\]

\[
= \chi(\text{maj}, D_m) - \frac{j + 1}{2}.
\]

Therefore \( \Delta(\text{maj}, D_n) = \Delta(\text{inv}, D_n) \) and so \( \nu(\text{maj}, D_n) = \tau(n). \)

We end this section by presenting a theorem which states that the set of inversion, descent, ascent and major index descendants of the dihedral group \( D_n \) coincides.

**Theorem 5.7.** For each \( n \geq 3 \) we have \( \Delta(\text{asc}, D_n) = \Delta(\text{des}, D_n) = \Delta(\text{inv}, D_n) = \Delta(\text{maj}, D_n). \)
References


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