Brauer-Fitting Correspondence on Tensor Algebra

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Abstract

We show that Brauer-Fitting correspondence as well as the notion of pseudo-blocks of endomorphism algebras are both compatible with the external tensor product of modules and algebras.

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1. Introduction

The tensor product of algebras and modules has been a subject for investigation by many authors in the past few decades. Külshammer [9] studied the tensor product of algebra and modules over an algebraically closed field. His results were generalized by M. Harris [2] who considered the tensor product of lattices and modules over local rings. Alghamdi and Khammash [1] investigated the tensor product of G -algebras as well as tensor product of the Brauer homomorphisms. Khammash [7] studied the points, pointed groups and the notion of the defect group in tensor product of G -algebras. W. Huang [3] analyzed the connection between block algebras and interior G -algebras in connection with the tensor product.

The concept of pseudo-blocks for endomorphism algebras was introduced in [8] where it was also shown (Theorem 2) that the distribution of the indecomposable summand of a module among the pseudo-blocks is compatible with the (Brauer) block distribution of the corresponding (under Brauer-Fitting correspondence (see section 1)) irreducible representations for the endomorphism algebra of that module. In this paper we relate the Brauer-Fitting correspondence as well as the notion of pseudo-blocks to the external tensor product of modules and algebras.

First we show that the Brauer-Fitting correspondence is compatible with the external tensoring of modules and algebras in the following sense

Theorem 1:

Suppose that Λ_1, Λ_2 are two finite dimensional algebras, and Y_i is Λ_i -module; $i=1,2$. If $X_i \in \text{Inds}(Y_i)$ having Brauer-Fitting correspondents $S_i \in \text{Irr}(E(Y_i))$, then $X_1 \otimes X_2 \in \text{Inds}(Y_1 \otimes Y_2)$ with Brauer-Fitting correspondent $S_1 \otimes S_2 \in \text{Irr}(E(Y_1 \otimes Y_2))$.

Here $\text{Inds}(Y_i)$ denotes the set of isomorphism classes of indecomposable direct Λ_i -summands of Y_i , $E(Y_i) = \text{End}_{\Lambda_i}(Y_i)$; the endomorphism algebra of Y_i and $\text{Irr}(E(Y_i))$ is the set of isomorphism classes of irreducible representations of $E(Y_i)$. A proof of theorem1 will be given in section 2.

We also show that such tensor operation is compatible with the pseudo-block distribution for the members of $\text{Inds}(X)$. Namely if we use the notation $\approx_{\text{psd}(X)}$ to denote the pseudo-block linkage relation defined on $\text{Inds}(X)$ (see section 4), then we have

Theorem2:

Suppose that $X_1 \otimes X_2, X'_1 \otimes X'_2 \in \text{Inds}(Y_1 \otimes Y_2)$. Then $X_1 \otimes X_2 \approx_{\text{psd}(Y_1 \otimes Y_2)} X'_1 \otimes X'_2$ if and only if $X_1 \approx_{\text{psd}(Y_1)} X'_1 \wedge X_2 \approx_{\text{psd}(Y_2)} X'_2$.

As the (Brauer) block distribution of $\text{Irr}(E(Y_1 \otimes Y_2))$ is compatible with the pseudoblock distribution of $\text{Inds}(Y_1 \otimes Y_2)$ ([8], Theorem2), theorem2 implies at once the following

Corollary1:

If $X_1 \otimes X_2, X'_1 \otimes X'_2 \in \text{Inds}(Y_1 \otimes Y_2)$ having Brauer-Fitting correspondents $S_1 \otimes S_2, S'_1 \otimes S'_2 \in \text{Irr}(E(Y_1 \otimes Y_2))$; respectively. Then $S_1 \otimes S_2 \approx_{E(Y_1 \otimes Y_2)} S'_1 \otimes S'_2$ if and only if $X_1 \approx_{\text{psd}(Y_1)} X'_1 \wedge X_2 \approx_{\text{psd}(Y_2)} X'_2$.

Theorem2 can also be applied in counting the pseudo-blocks of $X \otimes Y$. In fact if we write $\text{Inds}(X) \setminus \approx_{\text{psd}(X)}$ for the set of conjugacy classes for the relation $\approx_{\text{psd}(X)}$ on $\text{Inds}(X)$ then we have

Corollary2:

$$|\text{Inds}(X \otimes Y) \setminus \approx_{\text{psd}(X \otimes Y)}| = |\text{Inds}(X) \setminus \approx_{\text{psd}(X)}| \times |\text{Inds}(Y) \setminus \approx_{\text{psd}(Y)}|. \quad \square$$

2. Preliminaries

We start by describing the Brauer-Fitting correspondence. Let Λ be a finite dimensional k -algebra where F is an algebraically closed field of characteristic $p > 0$. Let Y be a finite dimensional Λ -module and write $E(Y) = \text{End}_{\Lambda}(Y)$; the endomorphism algebra of Y . An early and direct (and in fact vital) connection between the representation theory of the endomorphism algebra $E(Y)$ and representation theory of Λ is provided by the Brauer-Fitting theorem (see [10], 1.4) which gives a bijection (named Brauer-Fitting correspondence) between the set $\text{Irr}(E(Y))$ of isomorphism classes of simple $E(Y)$ -modules and the set $\text{Inds}(Y)$ of isomorphism classes of indecomposable direct Λ -summands of Y . In fact, if Y_i is an indecomposable direct summand of Y , then $P_i := (Y_i, Y)_{\Lambda} = \text{Hom}_{\Lambda}(Y_i, Y)$ is projective indecomposable $E(Y)$ -module and the Brauer-Fitting correspondence is given by the following bijection

$$(1) \quad Y_i \longleftrightarrow S_i := P_i / rP_i$$

where r denotes the radical. The interplay between properties of the modules S_i , Y_i involved in the correspondence (1) can be found in the references [4], [5], [6] and [7].

3. Connection with Tensor Product

We now relate the Brauer-Fitting correspondence to the tensor product of modules and algebras over a field F . Suppose that Λ_1, Λ_2 are two finite dimensional F -algebras, and Y_i is Λ_i -module; $i=1,2$. We write $\otimes$ for $\otimes_F$ and treat $Y_1 \otimes Y_2$ as a left $\Lambda_1 \otimes \Lambda_2$ -module via the diagonal action $(a_1 \otimes a_2)(y_1 \otimes y_2) = a_1 y_1 \otimes a_2 y_2$. It is known (see [2], Lemma 1.4) that $E(Y_1 \otimes Y_2) \cong E(Y_1) \otimes E(Y_2)$ as F -algebras.

Lemma 1.1:

If Λ_1, Λ_2 are two finite dimensional F -algebras, X_i, Y_i are two Λ_i -modules; $i=1,2$ then

- (1) $(X_1, Y_1)_{\Lambda_1} \otimes (X_2, Y_2)_{\Lambda_2} \cong (X_1 \otimes X_2, Y_1 \otimes Y_2)_{\Lambda_1 \otimes \Lambda_2}$
- (2) $r(X_1 \otimes X_2, Y_1 \otimes Y_2) = [r((X_1, Y_1)) \otimes (X_2, Y_2)] \oplus [(X_1, Y_1) \otimes r((X_2, Y_2))]$

Proof: (1) The map $(f, g) \mapsto f \otimes g$ defines an F -balanced map from $(X_1, Y_1)_{\Lambda_1} \times (X_2, Y_2)_{\Lambda_2}$ to $(X_1 \otimes X_2, Y_1 \otimes Y_2)_{\Lambda_1 \otimes \Lambda_2}$ which gives rise to an F -isomorphism $(X_1, Y_1)_{\Lambda_1} \otimes (X_2, Y_2)_{\Lambda_2} \cong (X_1 \otimes X_2, Y_1 \otimes Y_2)_{\Lambda_1 \otimes \Lambda_2}$

(2) This is special case of ([2], Prop.2.1) using (1). $\square$

Proof of Theorem1:

From the description of the Brauer-Fitting correspondence in (1), it follows that the Brauer-Fitting correspondent of $X_1 \otimes X_2 \in \text{Inds}(Y_1 \otimes Y_2)$ is

$$(X_1 \otimes X_2, Y_1 \otimes Y_2) / r(X_1 \otimes X_2, Y_1 \otimes Y_2) \cong (X_1, Y_1) \otimes (X_2, Y_2) / r((X_1, Y_1) \otimes (X_2, Y_2))$$

By lemma 1.1(1)

$$= (X_1, Y_1) \otimes (X_2, Y_2) / [r((X_1, Y_1)) \otimes (X_2, Y_2)] \oplus [(X_1, Y_1) \otimes r((X_2, Y_2))]$$

By lemma 1.2(2)

$$\cong (X_1, Y_1) / r(X_1, Y_1) \otimes (X_2, Y_2) / r(X_2, Y_2) = S_1 \otimes S_2. \quad \square$$

4. Blocks of Tensor Algebra

Here we recall some facts about the blocks and projective indecomposable modules for the tensor algebras $\Lambda_1 \otimes \Lambda_2$ and their block distribution which will be needed latter.

Lemma2.1:

(1) Every projective indecomposable $\Lambda_1 \otimes \Lambda_2$ - module has the form $P \otimes Q$ where P is projective indecomposable Λ_1 -module and Q is projective indecomposable Λ_2 -module.

(2) If $P \otimes Q$ and $P' \otimes Q'$ are two projective indecomposable $\Lambda_1 \otimes \Lambda_2$ -modules, then $P \otimes Q \underset{\Lambda_1 \otimes \Lambda_2}{\approx} P' \otimes Q'$ if and only if $P \underset{\Lambda_1}{\approx} P' \wedge Q \underset{\Lambda_2}{\approx} Q'$.

Proof:

(1) It is known (See [2], Cor.3.3) that every primitive idempotent in the algebra $\Lambda_1 \otimes \Lambda_2$ is of the form $\alpha \otimes \beta$ where $\alpha \in \Lambda_1, \beta \in \Lambda_2$ are primitive idempotents.

(2) $P \otimes Q \underset{\Lambda_1 \otimes \Lambda_2}{\approx} P' \otimes Q'$ if and only if both projective indecomposable $\Lambda_1 \otimes \Lambda_2$ -modules $P \otimes Q, P' \otimes Q'$ lie in the same block of $\Lambda_1 \otimes \Lambda_2$, say $B_1 \otimes B_2$. But by ([3], Lemma 2.2) we conclude that P, P' belong to the block B_1 and Q, Q' belong to the block B_2 , hence $P \underset{\Lambda_1}{\approx} P' \wedge Q \underset{\Lambda_2}{\approx} Q'$. $\square$

5. Connection with Pseudoblocks

The notion of pseudoblocks was introduced in [7] in order to adapt the block distribution of the members in $Inds(X)$ with the block distribution of the set $Irr(E(X))$

Definition 5.1 ([7]):

Define the relation $\underset{psd(X)}{\approx}$ on the set $Inds(X) = \{X_1, X_2, \dots, X_n\}$ as follows:

If $X_i, X_j \in Inds(X)$ then $X_i \underset{psd(X)}{\approx} X_j$ iff $\exists i = i_1, \dots, j = i_f$ such that, for all t , either $Hom(X_t, X_{t+1}) \neq 0$ or $Hom(X_{t+1}, X_t) \neq 0$.

The relation $\underset{psd(X)}{\approx}$ is clearly an equivalence relation on the set $Inds(X)$ whose equivalence classes are called the pseudo-blocks of the endomorphism algebra $E(X)$.

We now state the proof of theorem2

Proof of Theorem 2:

We note first that $(X_i, Y_i), (X'_i, Y_i); i=1,2$ are projective indecomposable $E(Y_i)$ -modules and that $E(Y_1 \otimes Y_2) \cong E(Y_1) \otimes E(Y_2)$. From ([7]; Theorem2] we have

$$X_1 \underset{psd(Y_1)}{\approx} X'_1 \wedge X_2 \underset{psd(Y_2)}{\approx} X'_2 \Leftrightarrow (X_1, Y_1) \underset{E(Y_1)}{\approx} (X'_1, Y_1) \wedge (X_2, Y_2) \underset{E(Y_2)}{\approx} (X'_2, Y_2)$$

$$\Leftrightarrow (X_1, Y_1) \otimes (X_2, Y_2) \underset{E(Y_1 \otimes Y_2)}{\approx} (X'_1, Y_1) \otimes (X'_2, Y_2) \quad (\text{by lemma 2.1(1)})$$

$$\Leftrightarrow (X_1 \otimes X_2, Y_1 \otimes Y_2)_{E(Y_1 \otimes Y)} \approx (X'_1 \otimes X'_2, Y_1 \otimes Y_2) \quad (\text{by lemma 1.1(2)})$$

$$\Leftrightarrow S_1 \otimes S_2_{E(X \otimes Y)} \approx S'_1 \otimes S'_2. \quad \square$$

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