Monoids in Proximal Banach Spaces

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Abstract

Given the binary operation $\circ_\delta : 2^V \times 2^V \longrightarrow 2^V$ defined on a subset of 2^V in the proximal Banach space V , we prove that the monoid $S = 2^V (\circ_\delta)$ is regular, every right ideal $A \subset S$ and left ideal $B \subset S$ are proximal, every ideal in S is idempotent and S is simple. For $A_\circ, B_\circ, C_\circ \in 2^V (\circ_\delta)$, we also prove that if $((A_\circ \circ_\delta B_\circ) \circ_\delta C_\circ) \circ_\delta D_\circ \neq \emptyset$, then $D_\circ$ and at least one of the groupoids $A_\circ, B_\circ, C_\circ$ is proximal.

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1 Introduction

Monoids in proximal Banach spaces are introduced in this article, which is an outgrowth of recent work on proximity spaces [5, 9] and their applications [7, 8, 6]. A *monoid* is a semigroup with an identity element. A *Banach space* is a complete, normed linear space. In this work, a Banach space is a Čech closure space [10, §14, p. 237] endowed with a proximity relation. Results concerning the proximity of monoids in such spaces are given.

2 Preliminaries

Let V be a Banach space, $A, B \subset V$, $x, y \in V$. The Hausdorff distance $D(x, A)$ is defined by $D(x, A) = \inf \{\|x - y\| : y \in A\}$ and $\|x - y\|$ is the distance between vectors x and y . The Čech closure [10] of A (denoted by $\text{cl}A$) is defined by $\text{cl}A = \{x \in V : D(x, A) = 0\}$. The sets A and B are proximal (near) (denoted $A \delta B$), provided $\text{cl}A \cap \text{cl}B \neq \emptyset$ [3]. A Banach space endowed with the proximity relation δ is called a proximal Banach space (denoted by (V, δ)).

Let $A \subset V$ be a nonempty subset of V . A binary operation $\circ : A \times A \rightarrow A$ maps $A \times A$ into A , where $A \times A$ is the set of all ordered pairs of elements of A [2]. A groupoid in V is a system defined by a nonempty set $A \subset V$ together with a binary operation $\circ$ on A (denoted by $A_\circ$) such that $A_\circ \subseteq A$.

3 Main Results

Lemma 3.1. *In a proximal Banach space (V, δ) , let $A, B, C \subset V$ such that $B \subseteq (A \cup C)$. If $A \delta B$ and $B \delta C$, then $A \delta C$.*

Proof. Case 1: Let $B \subseteq A$, $A \delta B$ and $B \delta C$. From $B \delta C$, $\text{cl}B \cap \text{cl}C \neq \emptyset$. Since $B \subseteq A$, $\text{cl}B \subseteq \text{cl}A$. Consequently, $\text{cl}A \cap \text{cl}C \neq \emptyset$. Hence, $A \delta C$. The other cases are similar. $\square$

From Lemma 3.1, obtain

Theorem 3.1. *In a proximal Banach space (V, δ) , let $A_\circ, B_\circ, C_\circ \subset V$ be groupoids such that $B_\circ \subseteq (A_\circ \cup C_\circ)$. If $A_\circ \delta B_\circ$ and $B_\circ \delta C_\circ$, then $A_\circ \delta C_\circ$.*

Let 2^V be the set of all subsets of V , $A, B \in 2^V$, and let

$$\delta = \{(A, B) \in 2^V \times 2^V : \text{cl}A \cap \text{cl}B \neq \emptyset\}.$$

Consider the binary operation $\circ_\delta : 2^V \times 2^V \rightarrow 2^V$ defined on a subset of 2^V , where

$$A \circ_\delta B = \begin{cases} A \cap B, & \text{if } A \delta B, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is easily shown that $(2^V, \circ_\delta)$ is a commutative monoid such that $V \in 2^V$ is the identity element of the monoid. And $(2^V, \circ_\delta)$ is called a proximal monoid in V (briefly, $2^V(\circ_\delta)$).

The following Theorem from K. Iséki [4] (see also [1]) serves as a characterization of regular monoids in proximal Banach spaces.

Theorem 3.2. (Iséki [4]) *Any semigroup $S(\cdot)$ is regular if and only if $A \cdot B = A \cap B$ for every right ideal $A \subset S$ and left ideal $B \subset S$.*

Theorem 3.3. *Let $S = 2^V(\circ_\delta)$ be a monoid in a proximal Banach space (V, δ) . The following statements are equivalent.*

- 1° S is regular.
- 2° $A \circ_\delta B = A \cap B$ for every right ideal $A \subset S$ and left ideal $B \subset S$.
- 3° Every ideal in S is idempotent [4, Theorem 2, p. 676].
- 4° S is simple, i.e., S itself is the only ideal of S [2, p. 4].

Proof. We only prove $2 \Leftrightarrow 1$: Let $B, A \subset S$ be left and right ideals, respectively.

$$A \circ_\delta S \circ_\delta B = A \circ_\delta B \subseteq B \Rightarrow A \delta B \Rightarrow A \circ_\delta B = A \cap B \Leftrightarrow$$

from Theorem 3.2, S is regular. □

Lemma 3.2. *Let (V, δ) be a proximal Banach space, $A, B, C \in 2^V(\circ_\delta)$. If $(A \circ_\delta B) \circ_\delta C \neq \emptyset$, then $A \delta C$ or $B \delta C$.*

Proof. Let $(A \circ_\delta B) \circ_\delta C \neq \emptyset$. Then $(A \circ_\delta B) \circ_\delta C = (A \cap B) \cap C$. From the definition of $\circ_\delta$, $(A \cap B) \delta C$, i.e., $\text{cl}(A \cap B) \cap \text{cl}C \neq \emptyset$. Since $\text{cl}(A \cap B) \subseteq \text{cl}A$ or $\text{cl}(A \cap B) \subseteq \text{cl}B$, then $\text{cl}A \cap \text{cl}C \neq \emptyset$ or $\text{cl}B \cap \text{cl}C \neq \emptyset$. Hence $A \delta C$ or $B \delta C$. □

Theorem 3.4. *Let (V, δ) be a proximal Banach space, $A_\circ, B_\circ, C_\circ \in 2^V(\circ_\delta)$. If $(A_\circ \circ_\delta B_\circ) \circ_\delta C_\circ \neq \emptyset$, then $A_\circ \delta C_\circ$ or $B_\circ \delta C_\circ$.*

Proof. Immediate from Lemma 3.2. □

Lemma 3.3. *Let (V, δ) be a proximal Banach space, $A, B, C \in 2^V(\circ_\delta)$. If $((A \circ_\delta B) \circ_\delta C) \circ_\delta D \neq \emptyset$, then D and at least one of the sets A, B, C are proximal.*

Proof. Immediate from Lemma 3.2. □

From Lemma 3.3, obtain

Theorem 3.5. *Let (V, δ) be a proximal Banach space and $A_\circ, B_\circ, C_\circ \in 2^V(\circ_\delta)$. If $((A_\circ \circ_\delta B_\circ) \circ_\delta C_\circ) \circ_\delta D_\circ \neq \emptyset$, then $D_\circ$ and at least one of the groupoids $A_\circ, B_\circ, C_\circ$ is proximal.*

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