On Automatic Continuity of Linear Operators in Certain Classes of Non-Associative Topological Algebras

Youssef Tidli

Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Marhaz
Fes, Morocco

Copyright © 2014 Youssef Tidli. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we address the problem of automatic continuity of homomorphisms in some topological non associative $*$ -algebras. To do this, we introduce and study on a non-associative algebra equipped with an involution $*$ a notion we call $*$ -semi-simplicity. It is based on the study of certain sided ideals called $*$ -ideals.

Keywords: automatic continuity, linear operators, non-associative topological algebras

1- Introduction

In this work, we define for an algebra in involution $*$ $(A, *)$ a concept that we call $*$ semi- simplicity, it rests on the study of certain ideals. The interest thus is to restrict with a family of the ideals instead of considering all the left ideals. This concept of $*$ semi- simplicity will contribute also under investigation of the problem of the automatic continuity of the linear operators in the topological algebras. While being based on the fact that the property to be locally bounded, is more essential, for an Banach algebra, that locally convexity, we were interested in the p -normed algebras $(0 < p \leq 1)$ [4]. We propose too to generalize the results known in the case of the Banach algebras to the case of the complete p -normed non associative algebras, in particular the non commutative Jordan algebras. We then show that if A is a $*$ semi-simple p -normed non commutative

Jordan algebra, any surjective homomorphism (or with dense range) of an complete p -normed non commutative Jordan algebra on A is continuous.

2- Preliminaries

Throughout this work, the algebras considered are supposed to be complex, non associative, unital and not necessarily commutative; and $(0 < p \leq 1)$.

An algebra over $\mathbb{K}$ is a $\mathbb{K}$ -vector space A with a bilinear map $(x, y) \rightarrow x \cdot y$ of $A \times A$ into A . If this product is associative (resp commutative), we say that the algebra is associative (resp. commutative). An algebra A is a Jordan algebra if every $(x, y) \in A^2$ we have $xy = yx$ and $(x^2 y)x = x^2 y$. An algebra A is a non-commutative Jordan algebra (n.c Jordan algebra) if every $(x, y) \in A^2$ we have $x(yx) = (xy)x$ and $x^2(yx) = (x^2 y)x$.

Let A be an algebra, we denote by A^+ the algebra A equipped with its vector space structure and the product $\circ$ defined by : $x \circ y = 1/2 (xy + yx)$ for all $x, y \in A$. The product $\circ$ is called the Jordan product. $(A^+, +, \cdot, \circ)$ is Jordan algebra. Let A be a n.c Jordan algebra. Recall that a modular maximal ideal M of A is a maximal ideal as the quotient algebra A / M is unital. A is called strongly semi-simple if the intersection of all modular maximal ideals of A (denoted $s\text{-rad}(A)$) is reduced to zero. An normed Jordan algebra A is a Jordan algebra equipped by a norm $\| \cdot \|$ satisfying $\|xy\| \leq \|x\| \|y\|$ for every $(x, y) \in A^2$, if $(A, \|\cdot\|)$ is Banach space then A is called a Jordan Banach algebra. Let A be a Jordan Banach algebra with unit e , if $x \in A$ such that $\|x - e\| \leq 1$, then x is invertible. Let A an algebra of Jordan-Banach, one calls the spectral ray of, noted $r(a)$, the limit $\lim_{n \rightarrow +\infty} \|a^n\|^{1/n}$ if it exists. A $*$ -ring is a ring with a map $*$: $A \rightarrow A$ that is an involution. More precisely, $*$ is required to satisfy the following properties: $(x + y)^* = x^* + y^*$, $(xy)^* = y^* x^*$, $1^* = 1$, $(x^*)^* = x$ for all x, y in A . This is also called an involutive ring, and ring with involution. Elements such that $x^* = x$ are called self-adjoint.

Also, one can define $*$ -versions of algebraic objects, such as ideal and subring, with the requirement to be $*$ -invariant: $x \in I \Rightarrow x^* \in I$ and so on. A $*$ -algebra A is a $*$ -ring, with involution $*$, such that $(\lambda x)^* = \bar{\lambda} x^* \quad \forall \lambda \in \mathbb{K}, x \in A$. An algebra A is called simple if it has no proper ideals. An $*$ -algebra A is called $*$ -simple if it has no proper $*$ ideals.

3- Automatic Continuity

One of the first results on the automatic continuity of homomorphism to dense image is in Rickart:

Theorem 3.1 [2]

Let θ Is a homomorphism of a Banach algebra A on an strongly semi-simple Banach algebra B . if the image of θ is dense in B , then θ is automatically continuous.

Theorem 3.2 [3]

Let T a homomorphism of a Jordan-Banach algebra A on a Jordan-Banach algebra B . If B is strongly semi-simple and if the image of T is dense in B , then T is automatically continuous.

Of its share, A.R. Palacios generalized the two theorems with the case of the complete normed associative power algebras.

Theorem 3.3 [4]

Any homomorphism with dense image of an complete normed associative power algebras A on an strongly semi-simple complete normed associative power algebras is automatically continuous.

The notion of separating space characterizes the continuity of linear operator. The utility of separating space comes owing to the fact that a linear operator is bounded if, and only if, its separating space is reduced to the singleton $\{0\}$ ([6]).

Definition: 3.1

Let T a linear application of a complete p -normed space X in a complete p -normed space Y . Then, the separating space $\sigma(T)$ of Y is the subset of Y defined by:

$$\sigma(T) = \{y \in Y / \exists (x_n)_n \subset X : x_n \xrightarrow{\|\cdot\|_p} 0 \text{ et } T(x_n) \xrightarrow{\|\cdot\|_p} y\}.$$

Proposition 3.1

Let X and Y two complete p -normed space, then the separating space $\sigma(T)$ of any a linear application $T: X \rightarrow Y$ is a closed subspace of Y .

Proof.

Evidently $\sigma(T)$ is a subspace vector of Y . Let $(y_k)_k$ a sequence in $\sigma(T)$ converging to y of Y . we prove that $y \in \sigma(T)$. For every $k \in \mathbb{N}^*$, $y_k \in$

$\sigma(T)$. Then there is a sequence $(y_{k,n})_n \subset X$ such that: $x_{n,k} \xrightarrow{\|\cdot\|_p} 0$ and $T(x_{n,k}) \xrightarrow{\|\cdot\|_p} y_k$. Let us choose two sequence $(z_k)_k$ $(y_k)_k$ in X such that, for every $k \in \mathbb{N}^*$, $\|z_k\|_p < 1/k$ and $\|T(z_k) - y\|_p < 1/k$.

Then, we are; $z_k \xrightarrow{\|\cdot\|_p} 0$ and $T(z_k) - y_k \xrightarrow{\|\cdot\|_p} 0$.
 On other hand, for every $k \in \mathbb{N}^*$, we are: $T(z_k) - y = (T(z_k) - y_k) + (y_k - y)$.
 then, $z_k \xrightarrow{\|\cdot\|_p} 0$ et $T(z_k) - y \xrightarrow{\|\cdot\|_p} 0$. Consequently, $y \in \sigma(T)$ ■

Based on the fact that the property of being locally bounded, is more essential for a complete normed algebra, the local convexity and using specific techniques non-associative algebras, we establish a complete p -normed Jordan algebra is the continuous inverse. Next, we show that the closure of own proper ideal of a complete p -normed Jordan algebra A is a own proper ideal. In particular, any maximal ideal of A is closed.

Definition 3.2

A topological algebra A is called locally bounded if t.v.s A is locally bounded i.e. If it has a neighborhood U of 0 , bounded (i.e absorbed by every neighborhood V of 0 : $\lambda U \subset V$)

A necessary and sufficient condition for a t.v.s either locally bounded is given by the following theorem Rovelin:

Theorem 3.4 ([1])

A t.v.s is locally bounded if, and only if, its topology may be defined by a p -norm $\|\cdot\|_p$ where p is a fixed real number $0 < p \leq 1$.

As in the associative case, a characterization of locally bounded algebras complete unit is given by the following theorem:

Theorem 3.5[1]

Let A be a complete metrizable nc Jordan unital algebra. Then the following assertions are equivalent:

1) There exist on A an equivalent distance d define the topology of A and such that $d(x,y) \leq d(x,0).d(0,y)$.

2) A is locally bounded.

3) The $\|\cdot\|_p$ topology of A may be defined by a norm $\|\cdot\|_p$, $0 < p \leq 1$. Checking $\|xy\|_p \leq \|x\|_p \|y\|_p$ for all x, y and $\|e\|_p = 1$.

A topological algebra equipped with a p -norm checking $\|xy\|_p \leq \|x\|_p \|y\|_p$ will be called later p -normed algebra, it is clear that when $p = 1$, we find the notion of normed algebra.

Proposition 3.2

Let A be a complete p -normed Jordan algebra A unital with unit e , then any element x of A such that $\|x - e\|_p < 1$ is invertible.

Proof.

Consider the associated algebra $BL(A)$ of bounded linear maps A , it is a complete p -normed algebra for the norm $\|\cdot\|_p$ defined by:

If $I_p = \sup_{\|x\|=1} \|f(x)\|_p$ where $\|\cdot\|_p$ is the p -norm of the algebra A . Then, it

suffices to note that $Lx - L_{x-e} = L_x = I$, where I is the identity of A ■

Theorem 3.4

Let A be a complete p -normed Jordan algebra A , then the set $(L - \text{Inv}(A))$ of l -invertible elements of A is open from A .

Proof.

It is known that the set $\text{Inv}(BL(A))$ of invertible elements of $BL(A)$ is open [16]. Furthermore, A is metrisable and complete, so x is invertible if and only if L_x (resp. R_x) is invertible in $BL(A)$. Since $G_L: x \rightarrow L_x$ (resp. $R_L: x \rightarrow R_x$) is continuous and $L\text{-inv}_d(A) = G_L^{-1} = (\text{Inv}(BL(A)))$ (resp. $L\text{-inv}_g(A) = R_L^{-1} = (\text{Inv}(BL(A)))$), whereas $L\text{-inv}_d(A)$ is an open of A ■

Corollary 3.1

Any complete p -normed Jordan unital algebra A is continuous l -invertible.

Remark

If A is unital flexible and x is invertible in A , then $L_x^{-1}(e) = R_x^{-1}(e)$ is unique and denoted x^{-1} . As a result, the application $x \rightarrow x^{-1}$ to $L - \text{Inv}(A)$ in $L - \text{Inv}(A)$ is a homeomorphism.

Corollary 3.2

Let A be a complete p -normed Jordan algebra A , then the closure of any proper ideal of A is a proper ideal of A . In particular, any maximal ideal of A is closed.

Proof.

Let M be a maximal ideal of A , then $e \notin M$ and $M \subseteq \overline{M} \subseteq A$, so $M = \overline{M}$ or $A = \overline{M}$, then $e \in \overline{M}$, through $L\text{-inv}(A) \cap M \neq \emptyset$. Let $x \in L\text{-inv}(A) \cap M$, then $x L_x^{-1}(e) = e \in M$, which is impossible. So $M = \overline{M}$ ■

Definition 3.3

Let A an n.c Jordan $*$ -algebra. We called $*$ -radical of A , noted $\text{Rad}^*(A)$, the intersection of all $*$ -maximal ideals of A . A is called $*$ semi-simple if $\text{Rad}^*(A) = \{0\}$.

Proposition 3.3

Let A be a non-associative $*$ -algebra and M $*$ -maximal ideal which is not maximal. Then there exists a maximal ideal N of A such that $M = N \cap N^*$.

Proof.

As M is not maximum, there is a maximal ideal N of A such that $M \subset N$. Since $M^* = M \subset N^*$, where $M \subset N \cap N^*$. As $N \cap N^*$ is a $*$ -ideal of A , it follows therefore that $M = N \cap N^*$ ■

Proposition 3.4

Let A be a complete p -normed n.c Jordan $*$ -algebra A , then all $*$ -maximal $*$ -ideal M of A is closed.

Proof.

If M is a maximal ideal of A , then M is closed. Otherwise, there is a maximal ideal N of A such that $M = N \cap N^*$ (proposition 3.3). Since N (resp. N^*) is closed, it is deduced that M is closed in A . Let A be a complete p -normed n.c Jordan algebra A and $a \in A$, then the spectrum of a noted $\text{Sp}(a) = \{ \lambda \in \mathbb{C} : a - \lambda e \notin G_j(A) \}$ where $G_j(A)$ is the set of elements a invertible of A is a non-empty set of $\mathbb{C}$, indeed: $\text{Sp}_A(a) = \text{Sp}_{C(a)}(a)$ where $C(a)$ is an associative and commutative p -normed algebra. Moreover, if A is complete, then $\text{Sp}_A(a)$ is a compact of $\mathbb{C}$ ■

Proposition 3.5

Let T a linear application of a complete p -normed unital algebra A in a complete p -normed unital algebra B . then, if T is surjective, the separating space $\sigma(T)$ is proper ideal of Y .

Proof.

Let $b \in B$ and $y \in \sigma(T)$.

$y \in \sigma(T)$, there then there exists a sequence $(a_n)_n \subset A$ such that:

$a_n \xrightarrow{\|\cdot\|_p} 0$ et $T(a_n) \xrightarrow{\|\cdot\|_p} y$. Suppose that T is surjective, then there exists

$a \in A$ such that $T(a) = B$; Then, $a_n a \xrightarrow{\|\cdot\|_p} 0$ et $T(a_n a) = T(a_n)(T(a)) = T(a_n)b$
 $\xrightarrow{\|\cdot\|_p} yb$, Consequently $yb \in \sigma(T)$. Let us show that $\sigma(T)$ is a proper ideal of B . As A and B are unital, $T(e_A) = e_B$. For all, $a \in A$, $\text{Sp}(T(a)) \subset \text{Sp}(a)$. Then
 $r_B(T(a)) \leq r_A(a)$ $a \in A$. Let c an element of center of B , then
 $r_B(T(a)) \leq r_B(c - T(a)) + r_B(T(a)) \leq \|c - T(a)\|_p + \|T(a)\|_p$
 Suppose that $e_B \in \sigma(T)$. Then, there exists a $(a_n)_n \subset A$ such that:
 $a_n \xrightarrow{\|\cdot\|_p} 0$ et $T(a_n) \xrightarrow{\|\cdot\|_p} e_B$. As e_B is an element of center of B ,
 $r_B(T(e_A)) = r_B(e_B) \leq \|e_B - T(a_n)\|_p + \|T(a_n)\|_p \xrightarrow{\|\cdot\|_p} 0$

What contradicts the fact that $r_B(e_B) = 1$ ■

Proposition 3.6

Let A an *-simple non associative *-algebra which is not simple. Then, there exists a subalgebra simple unit I of A such that $A = I \oplus I^*$.

Proof.

Let I a proper ideal of A . $I \cap I^*$ is one *-ideal, therefore $I \cap I^* = \{0\}$ or $I \cap I^* = A$. If $I \cap I^* = A$ then $I = A$, which is absurd. From where $I \cap I^* = \{0\}$. There is also $I + I^*$ is one *-ideal, then $I + I^* = \{0\}$ or $I + I^* = A$. If $I + I^* = \{0\}$, then $I = \{0\}$, which contradicts the fact that I is *proper*. Therefore, $A = I \oplus I^*$.

Let J a ideal of A such as $J \subseteq I$. According to what precedes, $A = J \oplus J^*$. Let $i \in I$, then there exists $j, j' \in J$ such that $i = j + j'^*$. However $i - j = j'^* \in I \cap I^* = \{0\}$, from where $i = j$, therefore $I = J$. Consequently, I is a minimal ideal of A .

Let J an ideal of I , then J is an ideal of A . Indeed, let $a \in A$ and $j \in J$, then it exists i, i'^* such that $a = i + i'^*$. From where $aj = (i + i'^*)j = ij + i'^*j$. However, $i'^*j \in I^*I \subseteq I \cap I^* = \{0\}$, consequently $aj = ij \in J$. As I is a minimal ideal, then $J = \{0\}$ or $I = J$. Thus, I is simple subalgebra. On other hand, I is unital and if 1 indicates the unit of A , then there exists $e, e' \in I$ such that $1 = e + e'^*$. Let $x \in I$, we are: $x = x1 = xe + xe'^*$, but $x - xe = xe'^* \in I \cap I^* = \{0\}$, from where $x = xe$. In the same way, we checked that $x = xe$. Consequently, I is unital of unit e ■

Lemma 3.1

Let T a homomorphism of an complete p -normed n.c Jordan algebra of A in an complete p -normed n.c Jordan algebra B , for any element a in A , $r(T(a)) \leq r(a)$.

Proof.

Let A and C under maximum commutative associative subalgebra of A containing a (the existence of is proven by the lemma of Zorn). Let D under maximum commutative associative algebra containing $T(C)$ in B . Still Let us note per T the definite restriction of T of C in D . it is clear that $\text{Sp}_D((T(a))) \subseteq \text{Sp}_C(a)$. it thus results from it that:

$$r(T(a)) \text{Sp}_D \{|\lambda| \mid \lambda \in \text{Sp}_D((T(a)))\} \leq \text{Sup}\{|\lambda| \mid \lambda \in \text{Sp}_C(a)\} = r(a) \blacksquare$$

Proposition 3.7

Let T a homomorphism of an complete p -normed n.c Jordan algebra A on an complete p -normed n.c Jordan algebra B then, if B is simple and if T is surjective (or with dense range), T is continuous.

Proof.

Let $\sigma(T)$ the separating space of T in B which is simple, therefore $\sigma(T) = \{0\}$ or $\sigma(T) = B$. the last case is impossible, indeed: Let e the unit of B . we have: $1 = r(e) = r(e - T(a) + T(a))$. However $T(a)$ and $e - T(a)$ commute, therefore: $r(e - T(a) + T(a)) \leq r(e - T(a)) + r(T(a))$. From where, for any element a of A , $1 \leq \|e - T(a)\|_p + \|a\|_p$. If $e \in \sigma(T)$, then there exists a sequence $(a_n)_{n \geq 1}$ which converges to 0 in A such that the sequence $(T(a_n))_{n \geq 1}$ converges to e in B . but $1 \leq \|e - T(a)\|_p + \|a\|_p \rightarrow 0$, which impossible. From where, $\sigma(T) = \{0\}$. Consequently, T is continuous $\blacksquare$

Theorem 3.5

Let T a homomorphism of an complete p -normed n.c Jordan algebra A on an complete p -normed n.c Jordan $*$ algebra B then, if B is $*$ simple and if T is surjective (or with dense range), T is continuous.

Proof.

Let B is an algebra $*$ simple, there exists simple unital subalgebra I of B such that $B = I \oplus I^*$ (Proposition 3.6); following algebraic isomorphism: $B \approx B/I^*$, one deduces that I is a maximum ideal of B . From where I (resp; I^*) is closed in B . Consequently, I (resp; I^*) is a complete p -normed subalgebra. Let us consider: $\text{Pr}_1: B \rightarrow I$ (resp. $\text{Pr}_2: B \rightarrow I^*$) the canonical projection of B on I (resp. of B on I^*). Since Pr_1 (resp. Pr_2) is a continuous epimorphism, then according to the proposition (3.7) $\text{Pr}_1 \circ T$ (resp. $\text{Pr}_2 \circ T$) is continuous. Consequently, $T = (\text{Pr}_1 + \text{Pr}_2) \circ T = \text{Pr}_1 \circ T + \text{Pr}_2 \circ T$ is continuous $\blacksquare$

Theorem 3.6

Let T is a homomorphism of a complete p -normed n.c Jordan algebra A on a complete p -normed n.c Jordan algebra B . If B is $*$ -semi-simple and if T is surjective (or dense range), then T is continuous.

Proof.

Let M^* -maximal ideal of B and $Q : B \rightarrow B/M$ the canonical surjection. Since Q is continuous, it results that $Q \circ T$ is a dense range. In addition, B/M is a $*$ - simple complete p -normed n.c Jordan $*$ -algebra. By Theorem (3.5), $Q \circ T$ is continuous. As a result that, $\sigma(Q \circ T) = \{0\}$. Since $\sigma(Q \circ T) = \overline{Q(\sigma(T))}$ [6], we deduce that $\sigma(Q(T)) = \{0\}$. Hence, $\sigma(T) \subset M$. As M is arbitrary, then $\sigma(T) \subset \bigcap M$. However $\bigcap M = \text{Rad}^*(B) = \{0\}$, where $\sigma(T) = \{0\}$. Consequently, T is continuous ■

Corollary 3.3

Let $(A, \|\cdot\|_p)$ an complete p -normed $*$ semi-simple n.c Jordan algebra. Then, we have:

- i) All the complete p -normed on A are equivalent.
- ii) The involution $*$ is automatically continuous.

Proof.

- i) It is enough to apply the previous theorem to the Identity of A .
- ii) That is to say q the linear application of A in $\mathbb{R}^+$ defined by $q(x) = \Pi x^* \Pi_p (x \in A)$.

We checks easily that Q is a complete p -normed on A . And according i), Q is equivalent to $\|\cdot\|_p$ ■

References

- [1] H. Kemmoun, Théorèmes type Gelfand-Mazur et Kaplansky en théorie des algèbres non associatives, Thèse Doctoral, Uni. Mohamed V, Rabat (2000).
- [2] Palacios, An approach to Jordan-Banach algebras from the theory of non associative complete normed algebras, Ann, Sci. Univ. Blaise Pascal, Clermont-Ferrand II, Serie Math. Fasc. 27, (1991), 1- 57.
- [3] P. S. Putter and B. Yood, Banach Jordan $*$ -algebra, Proc. London. Math. Soc. (3), 41, (1980), 21-44. <http://dx.doi.org/10.1112/plms/s3-41.1.21>
- [4] Rickart, C. E, General theory of Banach algebra, New York: Van Nostrand 1960.
- [5] L. Oukhtite, A. Tajmouati, Y. Tidli, On the automatic continuity of the epimorphisms in $*$ algebras of Banach. IJMMS 2004: 22 1183-1187 (2002).

[6] A. M. Sinclair, Automatic Continuity of linear operators, London. Math. Ploughshare Reading Notes Series 21 (C.U.P., Cambridge 1976).
<http://dx.doi.org/10.1017/cbo9780511662355>

Received: November 25, 2014; Published: December 30, 2014