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Some Distributions of BCL$^+$ - Algebras

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Abstract

BCL/BCL$^+$ -algebras were introduced by Yonghong Liu [1, 2], which are more extensive class than BCK/BCI/BCH-algebras [9-11]. In this work we introduce the notion of the extended distribution law in BCL$^+$ -algebras, i.e., we study shows that the distribution rules like left (resp. right) distribution rule, left (resp. right) weakly distribution rule, left (resp. right) strongly distribution rule, etc. In particular, properties of distributive BCL$^+$ -algebras are investigated.

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1 Introduction

The basic laws of the set operations are the distributive laws. In its application, for example: every distributive lattice can be embedded in a Boolean lattice [**7**]; for another example: rough distributive lattice in rough algebras [**5**]. Because of this, it is useful to study distributive structure in abstract algebra.

In [**1**, **2**], we initiated the BCL-algebras and BCL$^+$-algebras. Later on, we introduced the partial orders of a BCL$^+$-algebra [**4**] and topological BCL$^+$ -algebras [**6**]. In 2013, Deena and Rodyna [**3**] extended the study of the deformation to characterize BCL-algebras. Deena [**8**] recently introduced the notion of soft BCL-algebras. This paper is a study on algebraic structure of BCL^{+}-algebra, which is used for the extended distributive law, that is left (resp. right) distribution rule, left (resp. right) weakly distribution rule, left (resp. right) strongly distribution rule, left (resp. right) quasi-distribution rules. We give some characterizations of distributive BCL^{+}-algebras. Here, the remainder of the paper is organized as follows:

Section 2 deals with some basic concept of BCL^{+}-algebras. Section 3, for BCL^{+}-algebras, we introduce the notion of extended distribution rules. For the extended reason, some basic properties of the distributive BCL^{+}-algebras are derived.

2 Preliminaries

For distributive BCL^{+}-algebras, we will need the following results:

Law 2.1. [7] A lattice L is called distributive if for all $x, y, z \in L$ it satisfies the following properties:

(L 1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

(L 2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 2.1. [2] An algebra $(X; *, 1)$ is called a BCL^{+}-algebra if it satisfies the following laws hold: for any $x, y, z \in X$.

(BCL^{+}-1) $x * x = 1$.

(BCL^{+}-2) $x * y = 1$ and $y * x = 1$ imply $x = y$.

(BCL^{+}-3) $((x * y) * z) * ((x * z) * y) = ((z * y) * x)$.

Theorem 2.1. [2] Assume that $(X; *, 1)$ is any a BCL^{+}-algebra. Then the following hold: for any $x, y, z \in X$.

(i) $(x * (x * y)) * y = 1$.

(ii) $x * 1 = x$ imply $x = 1$.

(iii) $((x * y) * (x * z)) * (z * y) = 1$.

(iv) (BCL^{+}-2) $x * y = 1$ and $y * x = 1$ imply $x = y$.

Theorem 2.2. [2] An algebra $(X; *, 1)$ is a BCL^{+}-algebra if and only if it satisfies the following conditions: for all $x, y, z \in X$.

(i) (BCL^{+}-1) $x * x = 1$.

(ii) (BCL^{+}-2) $x * y = 1$ and $y * x = 1$ imply $x = y$.

(iii) $(((x * y) * z) * ((x * z) * y)) * ((z * y) * x) = 1$.

(iv) $x * (1 * y) = x$.

Definition 2.2. [2] Suppose that $(X;*,1)$ is a BCL⁺-algebra, the ordered relation if $x \le y$ if and only if $x * y = 1$, for all $x, y \in X$, then $(X;\le)$ is partially ordered set and $(X;*,1)$ is an algebra of partially ordered relation.

Corollary 2.1. [2] Let every $x \in X$. Then 1(one) is maximal element in a BCL⁺-algebra $(X;*,1)$ such that integral $1 \le x$ imply $x = 1$.

Definition 2.3. [4] Let $(X;*,1)$ be a $\mathrm{BCL}^+_{\le}$-algebra, for all $x, y, z \in X$, then

($\mathrm{BCL}^+_{\le}$-1) $x \le x$.

($\mathrm{BCL}^+_{\le}$-2) If $x \le y$ and $y \le x$ then $x = y$.

($\mathrm{BCL}^+_{\le}$-3) $((x*y)*z)*((x*z)*y) \le (z*y)*x$.

Definition 2.3. [6] Let $(X;\circ,*,1)$ be a BCL^+_{D}-algebra with two binary operations $\circ$ and $*$ that satisfies the following properties: for any $x, y, z \in X$.

(BCL^+_{D}-1) An algebra $D(X) = (X;\circ)$ is a distributive algebra.

(BCL^+_{D}-2) An algebra $P(X) = (X;*,1)$ is a algebra such that
$g(x,y,z) = (x*y)*z$.

(BCL^+_{D}-3) $x*(y\circ z) = (x*y)\circ(x*z)$ (right weakly distribution rule).

(BCL^+_{D}-4) $(y\circ z)*x = (y*x)\circ(z*x)$ (left weakly distribution rule).

Theorem 2.3. [4] Let $(X;*,1)$ be a $\mathrm{BCL}^+_{\le}$-algebra. Then binary relation $\le$ is a partial order on X.

Theorem 2.4. [4] Suppose that $(X;*,1)$ be a BCL⁺-algebra, we have that

$$((x*y)*(x*z))*(z*y) = 1, \text{ for all } x, y, z \in X, \tag{2.1}$$

Where $(X;\le)$ is partially ordered set. Then $\mathrm{BCL}^+_{\le}$-algebra is $(X;*,1)$.

3 Distributive BCL⁺-algebras

For a BCL⁺-algebra, the distributive axiom is just a basic concept in nature. We have the following definitions.

Definition 3.1. Let D be a set and let $\circ$ be a binary operation on D. Then D be a distributive set and we say that $\circ$ be a distribution (In fact, it satisfies distributive law of set theory). Define the following conditions: for $x, y, z \in D$.

(D1) For each $x, 1 \in D$, $1 \circ x = x$, $x \circ x = 1$.

(D2) x^{-1} exists and for each $x, x^{-1} \in D$ such that $x^{-1} \circ x = 1$.

(D3) $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$ (right distribution rule).

(D4) $(y \circ z) \circ x = (y \circ x) \circ (z \circ x)$ (left distribution rule).

Definition 3.2. Let X be a nonempty set and let $\circ$ be a binary operation on X. If $\circ$ be a distributive manner, then we call $(X;\circ)$ be a distributive algebra.

Remark 3.1. The operations $\circ$ and $*$ apparent are usually called multiplication and the distributive representation of the induced by $(X;\circ)$. Furthermore, the left (resp. right) weakly distribution rules of $*$ with respect to $\circ$ is obvious.

The use of distributions to representing BCL^{+}-algebra is illustrated in the following example, this show that BCL_D^{+}-algebra is true:

Example 3.1. Let $X=\{0,a,b,c,1\}$. We define two binary operations $*$ and $\circ$ on X by Table **3.1** and Table **3.2**:

Table 3.1. BCL^{+} operation

$*$	**0**	***a***	***b***	***c***	**1**
0	1	0	0	0	0
a	a	1	1	c	1
b	b	a	1	c	1
c	c	b	c	1	1
1	0	1	1	1	1

Table 3.2. D(X) operation

$\circ$	**0**	***a***	***b***	***c***	**1**
0	1	0	0	0	0
a	0	1	1	1	1
b	0	1	1	1	1
c	0	a	b	1	c
1	0	1	1	1	1

Then $(X;\circ,*,1)$ is a BCL_D^{+}-algebra.

Definition 3.3. Let P be nonempty subsets of BCL_D^{+}-algebra X together with an associative binary operation $*$ defined on X. We say that X is left (resp. right) stable if $x * p \in P$ (resp. $p * x \in P$) for each $x \in X$ and $p \in P$.

Definition 3.4. Let P be nonempty subsets of BCL_D^{+}-algebra X. Then left (resp. right) outstretched set if it satisfies the following properties:

(P 1) P be left (resp. right) stable subset of $D(X)$.

(P 2) For any $x, y \in X$, $x * y \in P$ and $x \in P$ imply $y \in P$.

Lemma 3.1. Let $x \in P$ with P be the outstretched sets of BCL_D^{+}-algebra X. If $x \le y$, then $y \in P$.

Proof. By Definition **2.2**, the partially ordered relation $x \le y$ implies $x * y = 1 \in P$. It is easy to see that distributive $x \in P$ and apply Definition **3.4**(P 2). Thus $y \in P$, as required.

Example 3.2. Let P be nonempty subsets of BCL_D^+-algebra *X*. From Example **3.1**, we conclude, therefore, that $P=\{1, a\}$ is an outstretched set of *X*.

Definition 3.5. Let $(X;*,1)$ be a BCL$^+$-algebra with a binary operation $*$ that satisfies the following properties: for all $x, y, z \in X$.

(BCL$^+$-rs) $x*(y*z)=(x*y)*(x*z)$ (right strongly distribution rule).

(BCL$^+$-ls) $(y*z)*x=(y*x)*(z*x)$ (left strongly distribution rule).

Theorem 3.1. Suppose that $(X;*,1)$ be a BCL$^+$-algebra. Then the following are equivalent: for all $x, y, z \in X$.

(i) X be left (resp. right) strongly distributions.
(ii) $x*1=x$.
(iii) $x*y=y*x$.
(iv) (BCL$^+$-rs) $x*(y*z)=(x*y)*(x*z)$.
(v) (BCL$^+$-ls) $(y*z)*x=(y*x)*(z*x)$.

Proof. Assuming (i), we conclude that the product is direct, and we suppose that $x, y, z \in X$, by Definition **3.5**, we can write

$$x*(y*z)=(x*y)*(x*z). \tag{3.1}$$

$$(y*z)*x=(y*x)*(z*x). \tag{3.2}$$

In particular, we have

$$\begin{aligned}(x*1)*x&=(x*x)*(1*x)\\&=1*(1*x)\end{aligned} \tag{3.3}$$

$$\begin{aligned}x*(x*1)&=(x*x)*(x*1)\\&=1*(x*1)\end{aligned} \tag{3.4}$$

By $x*1=1*x$, such $x*1=x$, and (ii) is proved.

Now, assuming (ii), we may write that

$$\begin{aligned}(x*y)*(y*x)&=((x*y)*y)*((x*y)*x)\\&=((x*y)*(y*y))*((x*x)*(y*x))\\&=((x*y)*1)*(1*(y*x))\\&=(x*y)*(1*(y*x))\\&=(x*y)*((x*y)*(y*x))\\&=1*((x*y)*(y*x))\end{aligned} \tag{3.5}$$

We may also write that

$$\begin{aligned}(y*x)*(x*y) &= ((y*x)*x)*((y*x)*y) \\ &= ((y*x)*1)*(1*(x*y)) \\ &= (y*x)*(1*(x*y)) \\ &= (y*x)*((y*x)*(x*y)) \\ &= 1*((y*x)*(x*y))\end{aligned} \tag{3.6}$$

Observe that $(x*y)*(y*x)=(y*x)*(x*y)$, so $x*y=y*x$, proving (iii).

That (iii) implies (iv), by Definition **3.5**(BCL$^+$-ls), we write

$$\begin{aligned}x*(y*z) &= (y*z)*x \\ &= (y*x)*(z*x) \\ &= (x*y)*(x*z)\end{aligned} \tag{3.7}$$

and (iv) is proved.

That (iv) implies (v), we write

$$\begin{aligned}(y*z)*x &= x*(y*z) \\ &= (x*y)*(x*z) \\ &= (y*x)*(z*x)\end{aligned} \tag{3.8}$$

and (v) is proved.

Finally, the fact that (v) implies (i) is obvious.

Theorem 3.2. A BCL$^+$-algebra X be left (resp. right) strongly distribution rules if and only if

$$x*y=y*(x*y) \text{ for all } x,y\in X. \tag{3.9}$$

Proof. By Theorem **3.1**, we have

$$\begin{aligned}y*(x*y) &= (y*x)*1 \\ &= (x*y)*1 \\ &= x*y.\end{aligned} \tag{3.10}$$

as desired.

Theorem 3.3. Let $(X;*,1)$ be $\mathrm{BCL}^+_\le$-algebra. Suppose the following conditions hold: for all $x,y,z\in X$

($\mathrm{BCL}^+_\le$-rq) $x*(y*z)\le(x*y)*(x*z)$ (right quasi-distribution rule).

($\mathrm{BCL}^+_\le$-lq) $(x*y)*z\le(x*z)*(y*z)$ (left quasi-distribution rule).

Then we say that $(X;*,1)$ be left (resp. right) quasi-distribution rules $\mathrm{BCL}^+_\le$-algebra.

Proof. Apply Theorem **3.1**(iv) (BCL$^+$-rs) and Definition **2.3**($\mathrm{BCL}^+_\le$-1), we conclude that $1\le 1$. Thus $x*(y*z)\le(x*y)*(x*z)$, proving part ($\mathrm{BCL}^+_\le$-rq). Similarly, part ($\mathrm{BCL}^+_\le$-lq) is proved.

Example 3.3. Let $X = \{0, a, b\}$. We define a binary operation $*$ on X by Table **3.3**:

Table 3.3. BCL^{+} operation

$*$	**0**	***a***	***b***
0	1	0	0
a	a	1	0
b	b	a	1

It is not difficult to verify that X be the right quasi-distribution rules, but we conclude that the left quasi-distribution rule cannot exist, and by Theorem **3.1**(iii), that also conclude that the left (resp. right) strongly distribution rules cannot exist.

Corollary 3.1. Let $(X; *, 1)$ be left (resp. right) quasi-distribution rules BCL^{+}-algebra and let $x, y \in X$. Then

(i) $1 * (1 * x) = 1 * x$.

(ii) $1 * (x * y) = 1 * (y * x)$.

Proof. For part (i), we have

$$1 * (1 * x) \geq (1 * 1) * (1 * x) = 1 * x. \tag{3.11}$$

$$1 * (1 * x) \leq 1 * (1 * (1 * x)) = 1 * (1 * x) = 1 * x. \tag{3.12}$$

Thus $1 * (1 * x) = 1 * x$, as desired.

For part (ii), we have

$$\begin{aligned} 1 * (x * y) &= (1 * x) * (1 * y) \\ &= (1 * (1 * x)) * (1 * y) \\ &= (1 * (1 * y)) * (1 * x) \\ &= (1 * y) * (1 * x) \\ &= 1 * (y * x) \end{aligned} \tag{3.13}$$

The proof is now complete.

Proposition 3.1. Let $(X; \circ, *, 1)$ be a BCL_D^{+}-algebra and let $x, y \in X$. Then

$$x * y \leq x * (x \circ y). \tag{3.14}$$

Proof. Apply Definition **2.3**(BCL_D^{+}-3), we have

$$\begin{aligned} x * (x \circ y) &= (x * x) \circ (x * y) \\ &= 1 \circ (x * y) \\ &= x * y \end{aligned} \tag{3.15}$$

By $x * y \leq x * y$. Then $x * y \leq x * (x \circ y)$, as required.

Theorem 3.4. Let $(X; \circ, *, 1)$ be a BCL_D^{+}-algebra and let $x, y, z \in X$. Then

$$x * z \leq (x \circ y) * (y \circ z). \tag{3.16}$$

Proof. We have

$$\begin{aligned}(x\circ y)*(y\circ z)&=(x*(y\circ z))\circ(y*(y\circ z))\\&=((x*y)\circ(x*z))\circ(y*z)\\&=((x*y)\circ(y*z))\circ((x*z)\circ(y*z))\end{aligned}\tag{3.17}$$

We first try

$$(x*y)\circ(y*z)=(x\circ y)*(y\circ z),\tag{3.18}$$

where the interchange of both $*$ and $\circ$.
Next we try

$$\begin{aligned}(x*z)\circ(y*z)&=(x*z)\circ(z*y)\\&=(x\circ z)*(z\circ y)\\&=(x\circ y)*(y\circ z)\end{aligned}\tag{3.19}$$

where z and y can be used interchangeably, by Definition **2.3**($\mathrm{BCL}^{+}_{\le}$ -2).
By Definition **3.1**(D1), in the case that

$$(x\circ y)*(y\circ z)=1.\tag{3.20}$$

By Theorem **2.3** and Theorem **2.4** we see that

$$\begin{aligned}x*z&=((x*z)*1)*1\\&=((x*z)*(x*y))*(y*z)\\&=((x*y)*(x*z))*(z*y)\\&=1\end{aligned}\tag{3.21}$$

then $1\le 1$, as required.

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