Symmetry Identities Related to the Generalized Twisted $q$-Euler Polynomials with Weak Weight $\alpha$

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Abstract

In this paper, we study the symmetry for generalized twisted $q$-Euler numbers $E_{n,\chi,q,\zeta}^{(\alpha)}$ and polynomials $E_{n,\chi,q,\zeta}^{(\alpha)}(x)$ with weak weight $\alpha$. We obtain some interesting identities of the power sums and generalized twisted $q$-Euler polynomials $E_{n,\chi,q,\zeta}^{(\alpha)}(x)$ using the symmetric properties for the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$.

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1 Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$(see [1-8]). Throughout this paper we use
the notation:
\[ [x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \]
Hence, \( \lim_{q \to 1} [x] = x \) for any \( x \) with \( |x|_p \leq 1 \) in the present \( p \)-adic case.

Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable function on \( \mathbb{Z}_p \). For \( g \in UD(\mathbb{Z}_p) \) the fermionic \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x) (-q)^x, \text{ see [1].} \tag{1.1}
\]

If we take \( g_n(x) = g(x + n) \) in (1.1), then we see that

\[
q^n I_q(g_n) + (-1)^{n-1} I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l). \tag{1.2}
\]

Note that
\[
\lim_{q \to 1} I_{-q}(g) = I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x).
\]

Let a fixed positive integer \( d \) with \( (p, d) = 1 \), set
\[
X = X_d = \lim_{N \to \infty} (\mathbb{Z}/dp^N \mathbb{Z}), \quad X_1 = \mathbb{Z}_p, \quad X^* = \bigcup_{0 < a < dp} a + dp\mathbb{Z}_p,
\]

\[
a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},
\]
where \( a \in \mathbb{Z} \) satisfies the condition \( 0 \leq a < dp^N \) (cf. [1, 4, 5, 8]).

Let \( T_p = \bigcup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N} \), where \( C_{p^N} = \{ \zeta^{p^N} = 1 \} \) is the cyclic group of order \( p^N \). For \( \zeta \in T_p \), we denote by \( \phi_{\zeta} : \mathbb{Z}_p \to \mathbb{C} \) the locally constant function \( x \mapsto \zeta^x \) (see [6, 8]). In [8], we introduced generalized twisted \( q \)-Euler numbers \( E_{n, \chi, q, \zeta}^{(\alpha)} \) and polynomials \( E_{n, \chi, q, \zeta}^{(\alpha)}(x) \) attached to \( \chi \). Let \( \chi \) be the primitive Dirichlet character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \) and \( \zeta \in T_p \). We assume that \( \alpha \in \mathbb{Z} \) and \( q \in \mathbb{C}_p \) with \( |q - 1|_p < 1 \). Let \( g(y) = \chi(y) \phi_{\zeta}(y) e^{(y + x)t} \). By (1.1), we derive

\[
\int_X \chi(y) \phi_{\zeta}(y) e^{(y+x)t} d\mu_{-q^t}(y) = \frac{[2]_q^\alpha \sum_{\alpha=0}^{d-1} \chi(a)(-1)^a \zeta^a q^{\alpha a} e^{at}}{\zeta^d q^{\alpha d} e^{\alpha t} + 1} e^{xt} \tag{1.3}
\]

By using Taylor series of \( e^{(y+x)t} \) in the above equation (1.3), we obtain

\[
\sum_{n=0}^{\infty} \left( \int_X \chi(y) \phi_{\zeta}(y) (y + x)^n d\mu_{-q^t}(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n, \chi, q, \zeta}^{(\alpha)}(x) \frac{t^n}{n!}.
\]
By comparing coefficients of \( \frac{t^n}{n!} \) in the above equation, we have the Witt formula for the generalized twisted \( q \)-Euler polynomials attached to \( \chi \) as follows:

**Theorem 1.1** For positive integers \( n \) and \( \zeta \in T_p \), we have

\[
E_{n,\chi,q,\zeta}^{(\alpha)}(x) = \int_X \chi(y) \phi_\zeta(y)(y + x)^n d\mu_{-q^\alpha}(y). \tag{1.4}
\]

Observe that for \( x = 0 \), the equation (1.4) reduces to (1.5).

**Corollary 1.2** For positive integers \( n \) and \( \zeta \in T_p \), we have

\[
E_{n,\chi,q,\zeta}^{(\alpha)} = \int_X \chi(y) y^n \phi_\zeta(y) d\mu_{-q^\alpha}(y). \tag{1.5}
\]

By (1.4) and (1.5), we have the following theorem.

**Theorem 1.3** For positive integers \( n \) and \( \zeta \in T_p \), we have

\[
E_{n,\chi,q,\zeta}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,\chi,q,\zeta}^{(\alpha)} x^{n-l}. \tag{1.6}
\]

## 2 Some identities for generalized twisted \( q \)-Euler polynomials with weak weight \( \alpha \)

In this section, we assume that \( q \in \mathbb{C}_p \) and \( \zeta \in T_p \). We obtain some interesting identities of the power sums and generalized twisted \( q \)-Euler polynomials \( E_{n,\chi,q,\zeta}^{(\alpha)}(x) \) using the symmetric properties for the \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \). If \( n \) is odd from (1.2), we obtain

\[
q^n I_q(g_n) + I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l g(l). \tag{2.1}
\]

Substituting \( g(x) = \chi(x) \zeta^x e^{xt} \) into the above, we obtain

\[
q^{and} \int_X \chi(x + nd) \zeta^{x+nd} e^{(x+nd)t} d\mu_{-q^\alpha}(x) + \int_X \chi(x) \zeta^x e^{xt} d\mu_{-q^\alpha}(x) = [2]_q^{nd} \sum_{j=0}^{n-1} (-1)^j \chi(j) \zeta^j q^{\alpha j} e^{jt}. \tag{2.2}
\]

For \( k \in \mathbb{Z}_+ \), let us define the \( p \)-adic functional \( T_{k,\chi,q,\zeta}^{(\alpha)}(n) \) as follows:

\[
T_{k,\chi,q,\zeta}^{(\alpha)}(n) = \sum_{l=0}^{n} (-1)^l \chi(l) q^{\alpha l} \zeta^l l^k. \tag{2.3}
\]
After some elementary calculations, we have

\[ q^{\alpha nd} \int_X \chi(x)\zeta^{x+nd}e^{(x+nd)t}d\mu_{-q^\alpha}(x) + \int_X \chi(x)\zeta^xe^{xt}d\mu_{-q^\alpha}(x) = (1 + \zeta^{d\alpha nd}e^{d\alpha dt}) \frac{[2]_{q^\alpha} \sum_{a=0}^{d-1} \chi(a)(-1)^{a\alpha q^\alpha e^{\alpha at}}}{\zeta^{d\alpha nd}e^{d\alpha dt} + 1}. \]

From the above, we get

\[ q^{\alpha nd} \int_X \chi(x)\zeta^{x+nd}e^{(x+nd)t}d\mu_{-q^\alpha}(x) + \int_X \chi(x)\zeta^xe^{xt}d\mu_{-q^\alpha}(x) = \frac{2}{\int_{Z_p} \zeta^{ndzq^{\alpha nd}e^{\alpha ndt}d\mu_{-1}(x)}}. \]

By (2.2), (2.3), and (2.4), we arrive at the following theorem:

**Theorem 2.1** Let \( n \) be odd positive integer. Then we obtain

\[ \frac{\int_X \chi(x)\zeta^xe^{xt}d\mu_{-q^\alpha}(x)}{\int_{Z_p} \zeta^{ndzq^{\alpha nd}e^{\alpha ndt}d\mu_{-1}(x)}} = \sum_{m=0}^{\infty} \left( \frac{[2]_{q^\alpha} T^{(a)}_{m,q,\zeta}(nd - 1)}{2} \right) \frac{t^m}{m!}. \]

Let \( w_1 \) and \( w_2 \) be odd positive integers. By Theorem 2.1, and after some elementary calculations, we have the following theorem.

**Theorem 2.2** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have

\[ \frac{\int_X \chi(x)\zeta^{w_2x}e^{w_2xt}d\mu_{-q^{w_2}}(x)}{\int_{Z_p} \zeta^{w_1w_2dx}q^{w_1w_2dx}e^{w_1w_2dx}d\mu_{-1}(x)} = \frac{[2]_{q^{w_2}}}{2} \sum_{m=0}^{\infty} \left( T^{(w_2)}_{m,w_1w_2,q,\zeta}(w_1d - 1)w_2^m \right) \frac{t^m}{m!}. \]

Then we set

\[ S(w_1, w_2) = \frac{\int_X \int_X \chi(x_1)\chi(x_2)\zeta^{(w_1x_1+w_2x_2)}e^{(w_1x_1+w_2x_2+w_1w_2x)}d\mu_{-q^{w_2}}(x_1)d\mu_{-q^{w_2}}(x_2)}{\int_{Z_p} \zeta^{w_1w_2dx}q^{w_1w_2dx}e^{w_1w_2dx}d\mu_{-1}(x)}. \]

By \( S(w_1, w_2) \) and Theorem 2.2, after elementary calculations, we obtain

\[ S(w_1, w_2) = \left( \sum_{m=0}^{\infty} E_{m,w_1,w_2}^{(w_1)}(w_2x)w_1^m \frac{t^m}{m!} \right) \left( \frac{[2]_{q^{w_2}}}{2} \sum_{m=0}^{\infty} T^{(w_2)}_{m,w_1w_2,q,\zeta}(w_1d - 1)w_2^m \frac{t^m}{m!} \right). \]

By using Cauchy product in the above, we obtain

\[ S(w_1, w_2) = \sum_{m=0}^{\infty} \left( \frac{[2]_{q^{w_2}}}{2} \sum_{j=0}^{m} \binom{m}{j} E_{j,w_1,w_2}^{(w_1)}(w_2x)w_1^j T^{(w_2)}_{m-j,w_1w_2,q,\zeta}(w_1d - 1)w_2^{m-j} \right) \frac{t^m}{m!}. \]
From the symmetry of $S(w_1, w_2)$ in $w_1$ and $w_2$, we also see that $S(w_1, w_2)$

$$S(w_1, w_2) = \left( \int_X \chi(x_2) \xi^{w_2} e^{(w_2+w_1)x_2} d\mu_{-w_1}(x_2) \right) \left( \int_Z \chi(x_1) \xi^{w_1} e^{x_1w_1} d\mu_{-w_2}(x_1) \right) \left( \int_Z \chi(x_1) \xi^{w_1} e^{x_1w_1} d\mu_{-w_1}(x_1) \right) \left( \int_Z \chi(x_1) \xi^{w_1} e^{x_1w_1} d\mu_{-w_2}(x_1) \right).$$

Thus we obtain

$$S(w_1, w_2) = \sum_{m=0}^{\infty} \left[ \frac{[2]}{2} \sum_{j=0}^{m} \binom{m}{j} w_1^{m-j} w_2^j E_{w_1, w_2}^{(w_1)}(w_1x) T_{m-j, w_1}(w_2d-1)w_1^{-j} \right] \frac{t^m}{m!}.$$

Thus we arrive at the following theorem:

**Theorem 2.3** Let $w_1$ and $w_2$ be odd positive integers. Then we have

$$[2]_{q} w_1 \sum_{j=0}^{m} \binom{m}{j} w_1^{m-j} w_2^j E_{j, w_1, w_2}^{(w_1)}(w_1x) T_{m-j, w_1}(w_2d-1)w_1^{-j} = [2]_{q} w_2 \sum_{j=0}^{m} \binom{m}{j} w_1^j w_2^{m-j} E_{j, w_1, w_2}^{(w_1)}(w_2x) T_{m-j, w_1}(w_1d-1),$$

where $E_{j, w_1, w_2}^{(w_1)}(x)$ and $T_{m-j, w_1}(k)$ denote generalized twisted $q$-Euler polynomials with weak weight $w_1$ and $p$-adic functional, respectively.

By Theorem 1.3 and Theorem 2.3, we have the following corollary.

**Corollary 2.4** Let $w_1$ and $w_2$ be odd positive integers. Then we obtain

$$[2]_{q} w_1 \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j E_{k, w_1, w_2}^{(w_1)}(w_2d-1)x^{j-k} = [2]_{q} w_2 \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^j w_2^{m-j} E_{k, w_1, w_2}^{(w_1)}(w_1d-1)x^{j-k}.$$

If we take $x = 0$ in Theorem 2.3, we also derive the interesting identity for generalized twisted $q$-Euler numbers with weak weight as follows:

**Corollary 2.5** Let $w_1$ and $w_2$ be odd positive integers. Then we obtain

$$[2]_{q} w_1 \sum_{j=0}^{m} \binom{m}{j} w_1^{m-j} w_2^j E_{j, w_1, w_2}^{(w_1)}(w_2d-1) = [2]_{q} w_2 \sum_{j=0}^{m} \binom{m}{j} w_1^j w_2^{m-j} E_{j, w_1, w_2}^{(w_1)}(w_1d-1).$$
By substituting Taylor series of $e^{x t}$ into (2.2), we obtain

$$
\sum_{m=0}^{\infty} \left( q^{a n d} \zeta^{n d} \int_{X} \chi(x + n d) \zeta^{x} (x + n d)^{m} d\mu_{-q^{a}} (x) + \int_{X} \chi(x) \zeta^{x} x^{m} d\mu_{-q^{a}} (x) \right) \frac{t^{m}}{m!} = \sum_{m=0}^{\infty} \left( [2] q^{n d} \sum_{j=0}^{n d-1} (-1)^{j} \chi(j) \zeta^{j} q^{a j} j^{m} \right) \frac{t^{m}}{m!}
$$

By comparing coefficients $\frac{t^{m}}{m!}$ in the above equation and (2.3), we have the following theorem.

**Theorem 2.6** Let $\chi$ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod} \; 2)$ and $\zeta \in T_{p}$. Then we have

$$
T^{(\alpha)}_{m, \chi, q, \zeta} (n d - 1) = q^{a n d} \zeta^{n d} E^{(\alpha)}_{m, \chi, q, \zeta} (n d) + E^{(\alpha)}_{m, \chi, q, \zeta} \left( \frac{2}{q^{a d}} \right).
$$

**References**


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