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On Green's Relations for Γ-semigroups and Reductive Γ-semigroups[1]

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Abstract

The notion of a Γ-semigroup has been introduced by M. K. Sen in the year 1981. In this paper, we consider Green's relations for Γ-semigroups and reductive Γ-semigroups.

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1 Introduction

The notion of a Γ-semigroup has been introduced by M. K. Sen in [6] the year 1981. Many classical notions of semigroup have been extended to Γ-semigroup (see [6], [7], [1], [2]). Green's relations for semigroups, first studied by J. A. Green [4], have played a fundamental role in the development to semigroup theory (see [5]). In [8], G. Thierrin has introduced a reductive semigroup. A. Fattahi and H. R. E. Vishki have given a characterization for regular reductive semigroups in [3]. In this paper, we consider Green's relations for Γ-semigroups and reductive Γ-semigroups. Moreover, we give a characterization for regular reductive Γ-semigroups.

2 Preliminaries

Let S and Γ be nonempty sets. If there exists a mapping $S \times \Gamma \times S \to S$, written (a, γ, b) by $a\gamma b$, S is called a Γ-*semigroup* if S satisfies the identities

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$(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.

Let S be an arbitrary semigroup and Γ be any nonempty set. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that S is a Γ-semigroup. Thus a semigroup can be considered to be a Γ-semigroup.

Let S be a Γ-semigroup and α be a fixed element in Γ. We define $a \cdot b = a\alpha b$ for all $a, b \in S$. We can show that $(S, \cdot)$ is a semigroup and we denote this semigroup by S_α.

An element a of a Γ-semigroup S is said to be *regular* if there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. A *regular* Γ*-semigroup* is a Γ-semigroup each element of which is regular. Let a be an element of a Γ-semigroup S and $\alpha, \beta \in \Gamma$. An element b of S is called an (α, β)-inverse of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$.

Let S be a Γ-semigroup and $\gamma \in \Gamma$. An element e in S is said to be a γ-idempotent if $e\gamma e = e$. The set of all γ-idempotents is denoted by E_γ. We denote $\bigcup_{\gamma \in \Gamma} E_\gamma$ by $E(S)$. The elements of $E(S)$ are called idempotent elements of S. A Γ-semigroup S is called an idempotent Γ-semigroup if $S = E(S)$.

3 Main results

The Green's equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ on a Γ-semigroup S are defined by the following rules :

(i) $a\mathcal{L}b$ if and only if $S^1\Gamma a = S^1\Gamma b$ where $S^1\Gamma a = S\Gamma a \cup \{a\}$.

(ii) $a\mathcal{R}b$ if and only if $a\Gamma S^1 = b\Gamma S^1$ where $a\Gamma S^1 = a\Gamma S \cup \{a\}$.

(iii) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

(iv) $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$.

Remark We have

(i) $a\mathcal{L}b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$.

(ii) $a\mathcal{R}b$ if and only if if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = b\alpha x$ and $b = a\beta y$.

(iii) $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$.

(iv) $a\mathcal{D}b$ if and only if there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$.

Theorem 3.1 $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

Proof. Let $(a, b) \in \mathcal{L} \circ \mathcal{R}$. Then there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$.

Case 1: $a = c$. Then $a\mathcal{R}b$. Since $a\mathcal{L}a$ and $a\mathcal{R}b$, $(a, b) \in \mathcal{L} \circ \mathcal{R}$.

Case 2: $b = c$. Then $a\mathcal{L}b$. Since $a\mathcal{L}b$ and $b\mathcal{R}b$, $(a, b) \in \mathcal{L} \circ \mathcal{R}$.

Case 3: $a \neq c$ and $b \neq c$. Since $a\mathcal{L}c$ and $c\mathcal{R}b$, there exist $x, y, u, v \in S$ and $\gamma, \mu, \eta, \theta \in \Gamma$ such that

$$x\gamma a = c, \qquad y\mu c = a, \qquad c\eta u = b, \qquad b\theta v = c.$$

Let $d = y\mu c\eta u$. Then

$$a\eta u = y\mu c\eta u = d$$

and

$$d\theta v = y\mu c\eta u\theta v = y\mu b\theta v = y\mu c = a$$

from which it follows $a\mathcal{R}d$. Also,

$$y\mu b = y\mu c\eta u = d$$

and

$$x\gamma d = x\gamma y\mu c\eta u = x\gamma a\eta u = c\eta u = b,$$

so $d\mathcal{L}b$. We deduce that $(a, b) \in \mathcal{R} \circ \mathcal{L}$. Therefore $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$.

Similarly, we can prove that $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$. ■

The $\mathcal{L}$-class (resp. $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class) containing the element a will be written L_a (resp. R_a, H_a, D_a).

Theorem 3.2 *Let* S *be a* Γ*-semigroup,* $\alpha \in \Gamma$ *and* e *be an* α*-idempotent. Then*

(i) $a\alpha e = a$ *for all* $a \in L_e$*.*

(ii) $e\alpha a = a$ *for all* $a \in R_e$*.*

(iii) $a\alpha e = a = e\alpha a$ *for all* $a \in H_e$*.*

(iv) *For all* $a \in S$, $|H_a \cap E_\alpha| \leq 1$.

Proof. (i) Let $a \in L_e$. Then $a\mathcal{L}e$. It follows that $S^1\Gamma a = S^1\Gamma e$. Then $a = e$ or there exist $x \in S$ and $\gamma \in \Gamma$ such that $a = x\gamma e$. If $a = e$, then $a\alpha e = e\alpha e = e = a$. If $a = x\gamma e$, then $a\alpha e = (x\gamma e)\alpha e = x\gamma(e\alpha e) = x\gamma e = a$.

(ii) It is similar to (i).

(iii) It follows from (i) and (ii).

(iv) Let $e, f \in H_a \cap E_\alpha$. Then $e\mathcal{H}f$. So $e\mathcal{L}f$ and $e\mathcal{R}f$. Then $f \in L_e$ and $e \in R_f$. By (i) and (ii), respectively , we have $f\alpha e = f$ and $f\alpha e = e$. Therefore $e = f$. It follows that $|H_a \cap E_\alpha| \leq 1$. ■

Theorem 3.3 *If* a *is a regular element of a* Γ*-semigroup* S*, then every element of* D_a *is regular.*

Proof. Since a is regular, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Let $b \in D_a$. So $a\mathcal{D}b$. Then $a\mathcal{L}c$ and $c\mathcal{R}b$ for some $c \in S$. Since $a\mathcal{L}c$, $a = c$ or there exist $u, v \in S$ and $\gamma, \mu \in \Gamma$ such that

$$u\gamma a = c \text{ and } v\mu c = a$$

Since $c\mathcal{R}b$, $b = c$ or there exist $z, t \in S$ and $\eta, \theta \in \Gamma$ such that

$$c\eta z = b \text{ and } b\theta t = c.$$

Case 1: $a = c$ and $c = b$. Then $a = b$, so b is regular.
Case 2: $a = c$ and $c\eta z = b$ and $b\theta t = c$.Then

$$b\theta(t\alpha x)\beta b = c\alpha x\beta c\eta z = a\alpha x\beta a\eta z = a\eta z = c\eta z = b$$

Case 3: $u\gamma a = c$ and $v\mu c = a$, and $b = c$.Then

$$b\alpha(x\beta v)\mu b = c\alpha x\beta v\mu b = u\gamma a\alpha x\beta a = u\gamma a = c = b$$

Case 4: $u\gamma a = c$ and $v\mu c = a$, and $c\eta z = b$ and $b\theta t = c$.Then

$$b\theta(t\alpha x\beta v)\mu b = c\alpha x\beta v\mu c\eta z = u\gamma a\alpha x\beta a\eta z = u\gamma a\eta z = c\eta z = b.$$

Therefore b is a regular element. ■

Let D is a $\mathcal{D}$-class. Then either every element of D is regular or no element of D is regular. We call the $\mathcal{D}$-class *regular* if all its elements are regular.

Theorem 3.4 *In a regular $\mathcal{D}$-class, each $\mathcal{L}$-class and each $\mathcal{R}$-class contains at least one idempotent.*

Proof. Let a be an element of a regular $\mathcal{D}$-class D in a Γ-semigroup S. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Then $x\beta a = x\beta(a\alpha x\beta a) = (x\beta a)\alpha(x\beta a)$. Thus $x\beta a$ is an α-idempotent. Since $a = a\alpha(x\beta a)$, $a\mathcal{L}x\beta a$. Similarly, $a\alpha x$ is a β-idempotent and $a\mathcal{R}a\alpha x$. ■

Theorem 3.5 *Let a be an element of a regular $\mathcal{D}$-class D in a Γ-semigroup S. Then*

(i) If a' is an (α, β)-inverse of a, then $a' \in D$ and the two $\mathcal{H}$-classes $R_a \cap L_{a'}$ and $L_a \cap R_{a'}$, contain a β-idempotent $a\alpha a'$ and an α-idempotent $a'\beta a$, respectively.

(ii) If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain a β-idempotent e and an α-idempotent f, respectively, then H_b contains an (α, β)-inverse a^* of a such that $a\alpha a^* = e$ and $a^*\beta a = f$.

(iii) No $\mathcal{H}$-class contains more than one (α, β)-inverse of a for all ordered pair $(\alpha, \beta) \in \Gamma \times \Gamma$.

Proof. (i) Let a' be an (α,β)-inverse of a. Then $a = a\alpha a'\beta a$ and $a' = a'\beta a\alpha a'$. Thus

$$a \ \mathcal{L}\ a'\beta a, \qquad a\alpha a' \ \mathcal{R}\ a, \qquad a' \ \mathcal{L}\ a\alpha a', \qquad a'\beta a \ \mathcal{R}\ a'.$$

Thus $a'\mathcal{D}a$, $a\alpha a' \in R_a \cap L_{a'}$ and $a'\beta a \in L_a \cap R_{a'}$. Therefore $a' \in D$. Since $a = a\alpha a'\beta a, a'\beta a = a'\beta a\alpha a'\beta a$ and $a\alpha a' = a\alpha a'\beta a\alpha a'$. Therefore $a'\beta a$ is an α-idempotent and $a\alpha a'$ is a β-idempotent.

(ii) Since $a\mathcal{R}e$, by Theorem 3.2(ii), $e\beta a = a$. Similarly, from $a\mathcal{L}f$ we deduce that $a\alpha f = a$ by Theorem 3.2(i). Again from $a\mathcal{R}e$ it follows that $a = e$ or there exist $x \in S$ and $\gamma \in \Gamma$ such that $a\gamma x = e$.

Case 1 : $a = e$. Let $a^* = f\beta e$. Then

$$a\alpha a^*\beta a = a\alpha(f\beta e)\beta a = (a\alpha f)\beta(e\beta a) = a\beta a = e\beta a = a$$

and

$$a^*\beta a\alpha a^* = (f\beta e)\beta a\alpha(f\beta e) = f\beta(e\beta a)\alpha f\beta e = f\beta(a\alpha f)\beta e = f\beta(a\beta e) = f\beta e = a^*.$$

Then a^* is an (α,β)-inverse of a. Moreover

$$a\alpha a^* = a\alpha f\beta e = a\beta e = e\beta e = e.$$

Further, since $a\mathcal{L}f$, $a = f$ or $f = y\theta a$ for some $y \in S$ and $\theta \in \Gamma$. If $a = f$, then $a^*\beta a = f\beta e\beta a = e\beta e\beta e = e = f$. If $f = y\theta a$, then $a^*\beta a = f\beta e\beta a = y\theta a\beta e\beta e = y\theta a = f$. It now follows easily that $a^* \in L_e \cap R_f = L_b \cap R_b = H_b$.

Case 2 : $a\gamma x = e$. Let $a^* = f\gamma x\beta e$. Then

$$a\alpha a^*\beta a = a\alpha(f\gamma x\beta e)\beta a = (a\alpha f)\gamma x\beta(e\beta a) = a\gamma x\beta a = e\beta a = a$$

and

$a^*\beta a\alpha a^* = (f\gamma x\beta e)\beta a\alpha(f\gamma x\beta e) = f\gamma x\beta(e\beta a)\alpha f\gamma x\beta e = f\gamma x\beta(a\alpha f)\gamma x\beta e = f\gamma x\beta(a\gamma x)\beta e = f\gamma x\beta e\beta e = f\gamma x\beta e = a^*$.

Then a^* is an (α,β)-inverse of a. Moreover

$$a\alpha a^* = a\alpha f\gamma x\beta e = a\gamma x\beta e = e\beta e = e.$$

Since $a\mathcal{L}f$, $a = f$ or there exist $y \in S$ and $\theta \in \Gamma$ such that $f = y\theta a$. If $a = f$, then $a^*\beta a = f\gamma x\beta e\beta a = a\gamma x\beta e\beta a = e\beta e\beta a = e\beta a = a = f$. If $f = y\theta a$, then $a^*\beta a = f\gamma x\beta e\beta a = y\theta(a\gamma x)\beta e\beta a = y\theta(e\beta e)\beta a = y\theta(e\beta a) = y\theta a = f$. It now follows easily that $a^* \in L_e \cap R_f = L_b \cap R_b = H_b$.

(iii) Suppose that a' and a^* are both (α,β)-inverses of a inside the single $\mathcal{H}$-class H_b. Since $a\alpha a'$ and $a\alpha a^*$ are β-idempotents in the $\mathcal{H}$-class $R_a \cap L_b$, $a\alpha a' = a\alpha a^*$ by Theorem 3.2(iv). Similarly, $a'\beta a = a^*\beta a$ because both are α-idempotents in the $\mathcal{H}$-class $L_a \cap R_b$. Then $a' = a'\beta a\alpha a' = a^*\beta a\alpha a^* = a^*$ ■

Let S be a Γ-semigroup. An equivalence relation ρ on S is called a *right [resp. left] congruence* on S if for each $a, b \in S$, $(a, b) \in \rho$ implies $(a\gamma t, b\gamma t) \in \rho$ [resp. $(t\gamma a, t\gamma b) \in \rho$] for all $t \in S$ and $\gamma \in \Gamma$. An equivalence relation ρ on S is called a *congruence* if ρ is both a right and left congruence. It is easy to prove that $\mathcal{L}$ is a right congruence on S and $\mathcal{R}$ is a left congruence on S.

Let S be a Γ-semigroup and ρ be a congruence on S. For $a\rho, b\rho \in S/\rho$ and $\gamma \in \Gamma$, let $(a\rho)\gamma(b\rho) = (a\gamma b)\rho$. This is well-defined, since for all $a, a', b, b' \in S$ and $\gamma \in \Gamma$,

$$\begin{aligned} a\rho = a'\rho \text{ and } b\rho = b'\rho &\Rightarrow (a, a'), (b, b') \in \rho \\ &\Rightarrow (a\gamma b, a'\gamma b), (a'\gamma b, a'\gamma b') \in \rho \\ &\Rightarrow (a\gamma b, a'\gamma b') \in \rho \\ &\Rightarrow (a\gamma b)\rho = (a'\gamma b')\rho. \end{aligned}$$

Let $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. We have

$$(a\rho\gamma b\rho)\mu c\rho = ((a\gamma b)\rho)\mu c\rho = ((a\gamma b)\mu c)\rho = (a\gamma(b\mu c))\rho = a\rho\gamma(b\mu c)\rho = a\rho\gamma(b\rho\mu c\rho).$$

Then the quotient set S/ρ is a Γ-semigroup.

Theorem 3.6 *Let S be a Γ-semigroup and ρ be a congruence on S. Then*
(i) If $\rho \subseteq \mathcal{L}$ then for all $a, b \in S$, $a\mathcal{L}b$ if and only if $a\rho\ \mathcal{L}\ b\rho$ in S/ρ.
(ii) If $\rho \subseteq \mathcal{R}$ then for all $a, b \in S$, $a\mathcal{R}b$ if and only if $a\rho\ \mathcal{R}\ b\rho$ in S/ρ.
(iii) If $\rho \subseteq \mathcal{H}$ then for all $a, b \in S$, $a\mathcal{H}b$ if and only if $a\rho\ \mathcal{H}\ b\rho$ in S/ρ.

Proof. (i) Let $a, b \in S$ such that $a\mathcal{L}b$. Then $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$.

Case 1 : $a = b$. Then $a\rho = b\rho$.

Case 2 : $a = x\alpha b$ and $b = y\beta a$. Then $a\rho = (x\alpha b)\rho = (x\rho)\alpha(b\rho)$ and $b\rho = (y\beta a)\rho = (y\rho)\beta(a\rho)$. Therefore $a\rho\mathcal{L}b\rho$.

Conversely, let $a, b \in S$. Assume $a\rho\mathcal{L}b\rho$. Then $a\rho = b\rho$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a\rho = (x\rho)\alpha(b\rho)$ and $b\rho = (y\rho)\beta(a\rho)$.

Case 1 : $a\rho = b\rho$. Then $(a, b) \in \rho$. Since $\rho \subseteq \mathcal{L}, (a, b) \in \mathcal{L}$. So $a\mathcal{L}b$.

Case 2 : $a\rho = (x\rho)\alpha(b\rho)$ and $b\rho = (y\rho)\beta(a\rho)$. Then $a\rho = (x\alpha b)\rho$ and $b\rho = (y\beta a)\rho$. Then $(a, x\alpha b) \in \rho$ and $(b, y\beta a) \in \rho$. Since $\rho \subseteq \mathcal{L}, (a, x\alpha b) \in \mathcal{L}$ and $(b, y\beta a) \in \mathcal{L}$. Then $a \in S^1\Gamma(x\alpha b)$ and $b \in S^1\Gamma(y\beta a)$. Thus $S^1\Gamma a = S^1\Gamma b$. Hence $a\mathcal{L}b$.

(ii) It is similar to (i).

(iii) It follows from (i) and (ii). ■

A congruence ρ on S is called *right [resp. left] reductive* if for each $a, b \in S$, $(a\gamma t, b\gamma t) \in \rho$ [resp. $(t\gamma a, t\gamma b) \in \rho$] for all $t \in S$ and $\gamma \in \Gamma$ implies $(a, b) \in \rho$. A Γ-semigroup S is called *right [resp. left] reductive* if equality on S is a right [resp. left] reductive congruence. In other words, S is called right [resp. left] reductive if for each $a, b \in S$, $a\gamma t = b\gamma t$ [resp. $t\gamma a = t\gamma b$] for all $t \in S$ and $\gamma \in \Gamma$ implies $a = b$. A Γ-semigroup is called *reductive* if it is both right and left reductive.

Theorem 3.7 *Let S be a Γ-semigroup and ρ be a congruence on S. The following statements are true.*

(i) ρ is a right reductive congruence if and only if S/ρ is a right reductive Γ-semigroup.

(ii) ρ is a left reductive congruence if and only if S/ρ is a left reductive Γ-semigroup.

(iii) ρ is a reductive congruence if and only if S/ρ is a reductive Γ-semigroup.

Proof. (i) Let ρ be a right reductive congruence. Let $a\rho, b\rho \in S/\rho$ such that $(a\rho)\gamma(t\rho) = (b\rho)\gamma(t\rho)$ for all $t \in S$ and $\gamma \in \Gamma$. Then $(a\gamma t, b\gamma t) \in \rho$ for all $t \in S$ and $\gamma \in \Gamma$. Since ρ is right reductive, $(a, b) \in \rho$. Hence $a\rho = b\rho$.

Conversely, suppose S/ρ is a right reductive Γ-semigroup. Let $a, b \in S$ such that $(a\gamma t, b\gamma t) \in \rho$ for all $t \in S$ and $\gamma \in \Gamma$. Then $(a\gamma t)\rho = (b\gamma t)\rho$ for all $t \in S$ and $\gamma \in \Gamma$. Thus $(a\rho)\gamma(t\rho) = (b\rho)\gamma(t\rho)$ for all $t \in S$ and $\gamma \in \Gamma$. Since S/ρ is a right reductive Γ-semigroup, $a\rho = b\rho$. Therefore $(a, b) \in \rho$.

(ii) It is similar to (i).

(iii) It follows by (i) and (ii). ■

Define two congruence ρ_r and ρ_l on a Γ-semigroup S as follows:

$$\rho_r = \{(a, b) \in S \times S \mid a\gamma t = b\gamma t \text{ for all } t \in S \text{ and for all } \gamma \in \Gamma\}$$
$$\rho_l = \{(a, b) \in S \times S \mid t\gamma a = t\gamma b \text{ for all } t \in S \text{ and for all } \gamma \in \Gamma\}.$$

The three following theorems hold.

Theorem 3.8 *Let S be a Γ-semigroup. Then*

(i) S is a right reductive Γ-semigroup if and only if $\rho_r = 1_S$.

(ii) S is a left reductive Γ-semigroup if and only if $\rho_l = 1_S$.

Proof. (i) Assume S is a right reductive Γ-semigroup. Let $a, b \in S$ such that $a\rho_r b$. Then $a\gamma t = b\gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. Since S is right reductive, $a = b$.

Conversely, suppose $\rho_r = 1_S$. Let $a, b \in S$ such that $a\gamma t = b\gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. Then $(a, b) \in \rho_r$. Since $\rho_r = 1_S$, $a = b$. Hence S is a right reductive Γ-semigroup.

(ii) It is similar to (i). ■

Theorem 3.9 *Let S be a regular Γ-semigroup. Then*

(i) $\rho_r \subseteq \mathcal{R}$.

(ii) $\rho_l \subseteq \mathcal{L}$.

Proof. (i) Let $(a,b) \in \rho_r$. Then $a\gamma t = b\gamma t$ for all $t \in S$ and for all $\gamma \in \Gamma$. So $a\Gamma S = b\Gamma S$. Since $a \in a\Gamma S$ and $b \in b\Gamma S$ because S is regular, $a\Gamma S^1 = b\Gamma S^1$. Therefore $(a,b) \in \mathcal{R}$. Thus $\rho_r \subseteq \mathcal{R}$.

(ii) It is similar to (i). ■

Theorem 3.10 *Let S be a regular Γ-semigroup. Then*
(i) ρ_r is the minimum right reductive congruence on S.
(ii) ρ_l is the minimum left reductive congruence on S.

Proof. (i) Let $a,b \in S$. Assume that $(a\gamma t, b\gamma t) \in \rho_r$ for all $t \in S$ and $\gamma \in \Gamma$. Then $a\gamma t\beta t' = b\gamma t\beta t'$ for all $t, t' \in S$ and $\gamma, \beta \in \Gamma$. Thus $a\alpha t'' = b\alpha t''$ for all $t'' \in S$ and $\alpha \in \Gamma$ because S is regular. So $(a,b) \in \rho_r$. Therefore ρ_r is a right reductive congruence on S.

Next, let ρ be any right reductive congruence on S. Let $(a,b) \in \rho_r$. Then $a\gamma t = b\gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. Since ρ is reflexive, $(a\gamma t, b\gamma t) \in \rho$. Therefore $(a,b) \in \rho$ because ρ is right reductive.

(ii) It is similar to (i). ■

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