International Journal of Algebra, Vol. 2, 2008, no. 1, 13 - 20

Rational Endomorphisms of a Nilpotent Group

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Abstract. Let G be a group. An endomorphism φ of G is called rational if there exist $a_1, \dots, a_r \in G$ and $h_1, \dots, h_r \in \mathbb{Z}$, such that $\varphi(x) = (x^{a_1})^{h_1} \dots (x^{a_r})^{h_r}$ for all $x \in G$. We denote by $End_r^*(G)$ the group of invertible rational endomorphisms of G. In this note, we prove that G is nilpotent of class c ($c \geq 3$) if and only if $End_r^*(G)$ is nilpotent of class $c-1$.

Mathematics Subject Classification: 20E36; 20E45

Keywords: Rational endomorphisms; nilpotent groups

1. **Introduction and results**

Let G be a group. A function $f : G \to G$ is said to be *rational* if there exist elements $a_1, \dots, a_r, a_{r+1} \in G$ and integers $h_1, \dots, h_r \in \mathbb{Z}$ such that

$$f(x) = a_1 x^{h_1} a_2 x^{h_2} \cdots a_r x^{h_r} a_{r+1} \text{ for all } x \in G.$$

If the exponents $h_1, \dots, h_r$ can be chosen to be positive, we say that f is *polynomial*; it is the case when for example the group G is of finite exponent. The set $F(G)$ of all rational functions of G forms a monoid with the usual composition of functions, this monoid has an important submonoid, $End_r(G)$ the set of rational functions which are endomorphisms. We denote by $F^*(G)$ and $End_r^*(G)$, the group of invertible elements of the corresponding monoids. Note that a bijective rational endomorphism of a group G is not necessarily in $End_r^*(G)$ (e.g., consider $G =]0, \infty[$ with multiplication and $f : x \mapsto x^3$). On the other hand, if G is finite, then $F^*(G) = Sym(G) \cap F(G)$ and $End_r^*(G) = Sym(G) \cap End_r(G)$, where $Sym(G)$ is the set of all bijections on G.

Following [3], we denote by $J(G)$ the set of integers m such that there exist t integers $m_1, \dots, m_t$ ($t \geq 1$) and t elements $c_1, \dots, c_t$ of G such that $m = m_1 + \cdots + m_t$ and $c_1 x^{m_1} \cdots c_t x^{m_t} = 1$ for all $x \in G$. The set $J(G)$ is an ideal of the ring of integers $\mathbb{Z}$ (see [3, p. 431]).

Let $m \geq 0$ be the integer such that $J(G) = (m)$ and $f \in F(G)$, where $f(x) = a_1 x^{h_1} \cdots a_r x^{h_r} a_{r+1}$. Put $h = h_1 + \cdots + h_r$ and $h_0 \in \{0, \dots, m-1\}$ is the unique integer such that $h_0 - h \in J(G)$. If $h_0 = h + qm$ for some integer q, then

$f(x) = f(x)(c_1x^{m_1}\cdots c_tx^{m_t})^q$. Thus we may assume that $h \in \{0,\dots,m-1\}$ and we call h the *degree* of f.
If $f \in End_r(G)$, then

$$f(x) = x^h \prod_{i=1}^{r}[x^{h_i}, b_i][x^{h_i}, b_i, x^{h-(h_1+\cdots+h_i)}],$$

where $b_i = (a_1\cdots a_i)^{-1}$ for all $i \in \{1,\dots,r\}$. (As G is not necessarily abelian, we should indicate that the above product is written in the increasing order, i.e., first the term for $i = 1$, then the one for $i = 2$, etc. In the remainder of paper, this convention will also be used).

Schweigert [5, Satz 3.5, p.37] showed that if G is a finite group, then G is nilpotent if and only if $End_r^*(G)$ is nilpotent. Corsi Tani and Rinaldi Bonafede [2, Theorem 3.5, p. 288] improved Schweigert's result by proving that if G is a finite group, then it is nilpotent of class $c \geq 2$ if and only if $End_r^*(G)$ is nilpotent of class $c-1$. Note that if G is abelian, then $End_r^*(G)$ coincides with the group of invertible universal power automorphisms which is contained in the center of $Aut(G)$, the automorphisms group of G. We generalize [2, Theorem 3.5, p. 288] for infinite groups, in fact we prove

Theorem 1.1. *Let G be a group and $c \geq 3$ be an integer. Then G is nilpotent of class c if and only if $End_r^*(G)$ is nilpotent of class $c-1$. Moreover, $End_r^*(G)$ is normal in $Aut(G)$.*

It was proved in [2, Theorem 5.2, p. 291] that if G is a finite group and $r \geq 2$ is a given integer, then the sum of any r inner automorphisms of G is an automorphism of G (i.e., the map $x \mapsto x^{a_1}\cdots x^{a_r}$ is an automorphisms of G for any r elements $a_1,\dots,a_r \in G$) if and only if G is 2-Engel with the property that $\binom{r}{2} \equiv 0 \mod exp(G')$ and $\gcd(r, \exp(G)) = 1$. Here we generalize this result to any group.

Let h be an integer. We say that a group G satisfies $\mathcal{I}(h)$ ($\mathcal{E}(h)$) if the map $x \longmapsto (x^{a_1})^{h_1}\dots(x^{a_r})^{h_r}$ is an invertible rational endomorphism (a rational epimorphism) of G for all $a_1, a_2,\dots,a_r \in G$ and $h_1, h_2,\dots,h_r \in \mathbb{Z}$ such that $h = h_1 + \cdots + h_r$.

Theorem 1.2. *Let G be a group and h an integer different of $-1, 0, 1$. Then:*

- *(i) G satisfies $\mathcal{I}(h)$ if and only if G is 2-Engel with conditions:*

  1. *$exp(G)$ is finite and $gcd(h, exp(G)) = 1$.*
  2. *$exp(G')$ divides $h(h-1)/2$.*

- *(ii) If G satisfies $\mathcal{E}(h)$, then G is a 2-Engel group with the only property that $exp(G')$ is finite and divides $h(h-1)/2$.*

If G admits a rational endomorphism $\varphi : x \mapsto (x^{a_1})^{h_1}(x^{a_2})^{h_2}(x^{a_3})^{h_3}$ such that $h_1 = h_2 = h_3 = 1$, G is called $3-$generalized abelian . It was proved that every 3−generalized abelian group is nilpotent of class at most 10 [[1], Theorem 3.1 p. 291]. Here we improve the bound if φ is an epimorphism of G.

Proposition 1.3. *If a group G admits an epimorphism of the form $x \mapsto x^{a_1}\ x^{a_2}\ x^{a_3}$, then G is nilpotent of class at most 8.*

2. **Proofs**

Our notation is standard. The commutator of n elements $x_1, x_2, \dots, x_n$ of a group G is defined inductively by
$[x_1] = x_1, [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and $[x_1, \cdots, x_{n-1}, x_n] = [[x_1, \cdots, x_{n-1}], x_n]$ $(n \geq 2)$. The $n\,th$ term of the lower central series is denoted by $\gamma_n(G)$ with $\gamma_1(G) = G$. We denote by $Z(G)$ the center of G.

To prove Theorem 1.1, we need the following lemma.

Lemma 2.1. *Let G be a group and let φ, ψ be two elements of $End_r^*(G)$ of degrees $\bar{h}$ and $\bar{k}$, respectively. Put*

$$\hat{\varphi}(x) = x^{-h}\varphi(x) = \prod_{i=1}^{r}[x^{h_i}, a_i][x^{h_i}, a_i, x^{h-(h_1+\cdots+h_i)}]$$

and

$$\hat{\psi}(x) = x^{-k}\psi(x) = \prod_{j=1}^{s}[x^{k_j}, b_j][x^{k_j}, b_j, x^{k-(k_1+\cdots+k_j)}].$$

Let l be an integer, $(l \geq 1)$, and assume that $\hat{\varphi}(x)$ and $\hat{\psi}(x)$ belong respectively to $\gamma_{l+1}(G)$ and $\gamma_2(G)$ for all $x \in G$. Suppose that $h = 1$ if $l \geq 2$. Then

- *(i)*

$$\hat{\psi}(\hat{\varphi}(x)) \equiv \hat{\varphi}(\hat{\psi}(x)) \equiv 1\, mod \gamma_{l+2}(G).$$

- *(ii)*

$$[\varphi, \psi] \equiv id_G \left\{ \begin{array}{c} mod\gamma_3(G)\ if\ l = 1,\ h \text{ and } k \text{ arbitrary.} \\ mod\gamma_{l+2}(G)\ \ if\ h = 1,\ (l \geqslant 2) \text{ and } k \text{ arbitrary.} \end{array} \right\}.$$

Proof. **(i)** Because

$$\hat{\psi}(\hat{\varphi}(x)) = \prod_{j=1}^{s}[(\hat{\varphi}(x))^{k_j}, b_j][(\hat{\varphi}(x))^{k_j}, b_j, (\hat{\varphi}(x))^{k-(k_1+\cdots+k_j)}] \equiv 1\, mod \gamma_{l+2}(G).$$

For $l = 1$, we prove, in the same manner, that

$$\hat{\varphi}(\hat{\psi}(x)) \equiv 1\, mod \gamma_{l+2}(G).$$

Assume that $l \geq 2$, then $h = 1$. We remark that

$$\hat{\varphi}(x_1^{\varepsilon_1}x_2^{\varepsilon_2}...x_t^{\varepsilon_t}) \equiv \hat{\varphi}(x_1)^{\varepsilon_1}\hat{\varphi}(x_2)^{\varepsilon_2}...\hat{\varphi}(x_t)^{\varepsilon_t}\ mod\gamma_{l+2}(G)$$

for all

$$x_1, x_2, ..., x_t \in G(\varepsilon_i = \pm 1),$$

because

$$\hat{\varphi}(x_1 x_2) = \hat{\varphi}(x_1)[\hat{\varphi}(x_1), x_2]\hat{\varphi}(x_2) \equiv \hat{\varphi}(x_1)\hat{\varphi}(x_2)$$

and

$$\hat{\varphi}(x_1^{-1}) = \hat{\varphi}(x_1)^{-1}[\hat{\varphi}(x_1)^{-1}, x_1^{-1}] \equiv \hat{\varphi}(x_1)^{-1}\, mod \gamma_{l+2}(G).$$

Then

$$\hat{\varphi}(\hat{\psi}(x)) = \hat{\varphi}(\prod_{j=1}^{s}[x^{k_j}, b_j][x^{k_j}, b_j, x^{k-(k_1+\cdots+k_j)}]) \equiv$$

$$\prod_{j=1}^{s}[\hat{\varphi}(x)^{k_j}, \hat{\varphi}(b_j)][\hat{\varphi}(x)^{k_j}, \hat{\varphi}(b_j), \hat{\varphi}(x)^{k-(k_1+\cdots+k_j)}] \equiv 1\, mod \gamma_{l+2}(G).$$

(ii) We now have $\varphi(\psi(x)) = \varphi(x^k\hat{\psi}(x)) = \varphi(x)^k\varphi(\hat{\psi}(x)) = (x^h(\hat{\varphi}(x))^k\hat{\psi}(x)^h\hat{\varphi}(\hat{\psi}(x))$.

In the same way $\psi(\varphi(x)) = (x^k\hat{\psi}(x))^h\hat{\varphi}(x)^k\hat{\psi}(\hat{\varphi}(x))$. It is easy to check by induction that $(x^h(\hat{\varphi}(x))^k \equiv x^{hk}\hat{\varphi}(x)^k \bmod \gamma_{l+2}(G)$,and $(x^k\hat{\psi}(x))^h \equiv x^{hk}\hat{\psi}(x)^h \bmod \gamma_3(G)$. Then

$$\varphi(\psi(x)) \equiv \begin{cases} x^{hk}\hat{\varphi}(x)^k\hat{\psi}(x)^h\, mod \gamma_3(G) \text{ if } l = 1 \\ x^k\hat{\varphi}(x)^k\hat{\psi}(x)\, mod \gamma_{l+2}(G) \text{ if } \ l \geq 2 \text{ and } h = 1\,. \end{cases}$$

and

$$\psi(\varphi(x)) \equiv \begin{cases} x^{hk}\hat{\psi}(x)^h\hat{\varphi}(x)^k\, mod \gamma_3(G), \text{ if } l = 1 \\ x^k\hat{\psi}(x)\hat{\varphi}(x)^k\, mod \gamma_{l+2}(G), \text{ if } l \geq 2 \text{ and } h = 1\,. \end{cases}$$

Therefore $\varphi(\psi(x)) = \psi(\varphi(x))\,(\psi(\varphi(x))^{-1}\varphi(\psi(x))) \equiv \psi(\varphi(x))\Delta(x)\, mod \gamma_{l+2}(G)$ where

$$\Delta(x) = \begin{cases} \left[\hat{\varphi}(x)^k, \hat{\psi}(x)^h\right] \equiv 1\ mod \gamma_3(G) \text{ if } l = 1 \\ \left[\hat{\varphi}(x)^k, \hat{\psi}(x)\right] \equiv 1\ mod \gamma_{l+2}(G) \text{ if } l \geq 2, \text{and } h = 1. \end{cases}$$

Hence $[\varphi, \psi]\,(x) = \varphi^{-1} \circ \psi^{-1} \circ (\varphi \circ \psi)(x) \equiv (\psi \circ \varphi)^{-1} \circ (\psi \circ \varphi)(x)$

$$\equiv x \begin{cases} mod \gamma_3(G), \text{if } l = 1, \\ mod \gamma_{l+2}(G), \text{if } l \geq 2, \text{and } h = 1. \end{cases}$$

This completes the proof. □

Lemma 2.2. *Let G be a group and $\varphi_1, \varphi_2, \ldots, \varphi_n \in End_r^*(G)$ $(n \geq 2)$. Then $[\varphi_1, \varphi_2, \ldots, \varphi_n](x) \equiv x\ mod \gamma_{n+1}(G)$ for all $x \in G$.*

Proof. We argue by induction on n. For $n = 2$, let φ and $\psi \in End_r^*(G)$, there exist $\hat{\varphi}(x)$ and $\hat{\psi}(x) \in \gamma_2(G)$ such that $\varphi(x) = x^h\hat{\varphi}(x)$ and $\psi(x) = x^k\hat{\psi}(x)$, for all $x \in G$, where h and k are the degrees of φ and ψ respectively.

By the previous lemma for $l = 1$, we have $[\varphi, \psi] \equiv id_G \bmod \gamma_3(G)$.

Now assume inductively that the lemma is true for n elements of $End_r^*(G)$ and let $\varphi_1, \dots, \varphi_n, \varphi_{n+1} \in End_r^*(G)$. Put $\varphi = [\varphi_1, \dots, \varphi_n]$ and $\psi = \varphi_{n+1}$. By induction, there exist $\hat{\varphi}(x)$ and $\hat{\psi}(x)$ belonging to $\gamma_{n+1}(G)$ and $\gamma_2(G)$ respectively, such that $\varphi(x) = x\hat{\varphi}(x)$ and $\psi(x) = x^k\hat{\psi}(x)$ for all $x \in G$. By the previous lemma, for $h = 1$ and $l = n \geq 2$, we may write $[\varphi_1, \dots, \varphi_n, \varphi_{n+1}] = [\phi, \psi] \equiv id_G \bmod \gamma_{n+2}(G)$. Which will end the induction. □

Proof of Theorem 1.1. Let c be a class of nilpotence of G ($c \geq 2$). Applying Lemma 2.2 for $n = c$, we have $\gamma_{c+1}(G) = \{1\}$. Then

$$[\varphi_1, \dots, \varphi_c] = \mathrm{Id}_G$$

for any $\varphi_1, \dots, \varphi_c \in \mathrm{End}_r^*(G)$. Thus $\mathrm{End}_r^*(G)$ is nilpotent of class at most $c-1$. Since $Inn(G) \leq End_r^*(G)$ is of class $c-1$, it follows that $End_r^*(G)$ so is. Conversely, if $End_r^*(G)$ is nilpotent of class $c-1$ ($c \geq 3$), $Inn(G)$ is of class at most $c-1$. Thus G is nilpotent of class at most c. If G is of class at most $c-1$, then by the first part, $End_r^*(G)$ is of class at most $c-2$, a contradiction. If $\mathrm{End}_r^*(G)$ is abelian, G is abelian or nilpotent of class 2, according to that $Inn(G)$ is trivial or not. This completes the proof. □

Lemma 2.3. *Let G be a group and n be an integer such that the map $x \mapsto x^n$ is an homomorphism of G. Then*

(i) $(xy)^{n-1} = y^{n-1}x^{n-1}$ for all $x, y \in G$.

(ii) Moreover, if this map is an epimorphism of G, the map $x \mapsto x^{n-1}$ is an homomorphism from G to $Z(G)$.

Proof. Let $x, y \in G$.

(i) $(xy)^{n-1} = ((yx)^y)^{n-1} = ((yx)^{n-1})^y = ((yx)^n x^{-1}y^{-1})^y = (y^n x^n x^{-1}y^{-1})^y = y^{n-1}x^{n-1}$

(ii) There exists $a \in G$ such that $y = a^n$. Thus $[x^{n-1}, y] = [x^{n-1}, a^n] = x^{1-n}a^{-n}x^{n-1}a^n = x(x^{-1})^n(a^{-1})^n x^n x^{-1}a^n x x^{-1} = x(x^{-1}a^{-1}x)^n(x^{-1}ax)^n x^{-1} = x(x^{-1}a^{-1}xx^{-1}ax)^n x^{-1} = 1$. It follows that $(xy)^{n-1} = y^{n-1}x^{n-1} = x^{n-1}y^{n-1}$. This completes the proof. □

Lemma 2.4. *Let G be a 2-Engel group. Then for all $x, y \in G$ and $m \in \mathbb{Z}$, we have*

(i) $[x^m, y] = [x, y^m] = [x, y]^m$.

(ii) $(xy)^m = x^m y^m [y, x]^{\frac{m(m-1)}{2}}$

(iii) Moreover, if the map $x \mapsto x^m$ ($m \neq, 0, 1$) is an endomorphism of G, then $exp(G')$ is finite and divides $m(m-1)/2$.

Proof. (i) and (ii) see [[4] Hilfssatz III,1.3].

(iii) It is obvious, because G' is abelian. □

Lemma 2.5. *Let G be a nilpotent group and $f \in End_r(G)$.*

$$\begin{aligned} f(x) &= (x^{a_1})^{h_1} \dots (x^{a_r})^{h_r} \\ &= x^h \prod_{i=1}^{r} [x^{h_i}, a_i][x^{h_i}, a_i, x^{h-(h_1+\cdots+h_i)}] \end{aligned}$$

with $h = h_1 + \cdots + h_r$. Then the following assertions are equivalent:
(i) $f \in End_r^*(G)$.
(ii) $\bar{h}$ *is an invertible element of* $\frac{\mathbb{Z}}{J(G)}$.
(iii) If G has finite exponent, then $\gcd(h, exp(G)) = 1$*; otherwise* $h \in \{1, -1\}$.

Proof. The proof is similar to that of [3, Theorem 1, p. 432], except (iii)⇒ (i). Endimioni has proved the implication $(iii) \Rightarrow (i)$, if $f \in F(G)$ [3, theorem 1, p. 432] and the reciprocal map of f is defined by $f^{-1}(x) = f'(x)g(x)^{-k}$ where f' and g satisfy $\bar{f} \circ \bar{f'} = \bar{f'} \circ \bar{f} = Id_{\frac{G}{Z(G)}}$, $g(G) \subseteq Z(G)$ and $f(f'(x)) = xg(x)$ for all $x \in G$, and k satisfies $hk \equiv 1 \pmod{\exp G}$ if G is of finite exponent, otherwise $k = h$. It is easy to check that f' is a rational map of G satisfying $f'(xy) \equiv f'(x)f'(y) \mod Z(G)$. Thus, $f^{-1}(x) = f'(x)\left(x^{-1}f\left(f'(x)\right)\right)^{-k}$. We now prove that f^{-1} is an endomorphism of G. Noting that f^{-1} is rational and $x^{-1}f((f'(x)) \in Z(G)$ for all $x \in G$, we have

$$f^{-1}(xy) = f'(xy)((xy)^{-1}f(f'(xy)))^k = f'(x)f'(y)z(y^{-1}x^{-1}f(f'(x)f'(y)z))^{-k}$$

for some $z \in Z(G)$. Thus

$$\begin{aligned} f^{-1}(xy) &= f'(x)f'(y)z(y^{-1}x^{-1}f(f'(x))f(f'(y))f(z))^{-k} \\ &= f'(x)f'(y)z(x^{-1}f(f'(x))y^{-1}f(f'(y))f(z))^{-k} \\ &= f'(x)(x^{-1}f(f('(y)))^{-k}f'(y)(y^{-1}f(f'(y)))^{-k}zf(z)^{-k} \end{aligned}$$

But since $f(z) = z^h$,

$$zf(z)^{-k} = z(z^h)^{-k} = zz^{-hk} = zz^{-1} = 1$$

Hence $f^{-1}(xy) = f^{-1}(x)f^{-1}(y)$ for all $x, y \in G$ and $f^{-1} \in End_r(G)$. □

Proof of Theorem 1.2.

(i) Suppose that G satisfies $\mathcal{I}(h)$. In particular, the maps $x \mapsto x^h$ and $x \mapsto x^{h-2}(x^a)^2$ are invertible rational endomorphisms of G. By Lemma 2.3, the map $x \mapsto x^{h-1}$ is an homomorphism from G to $Z(G)$. Then for all $x, y, a \in G$, we have

$$(xy)^{h-2}((xy)^a)^2 = x^{h-2}(x^a)^2y^{h-2}(y^a)^2.$$

Hence

$$\begin{aligned} (xy)^{h-1}y^{-1}x^{-1}x^ay^ax^ay^a &= x^{h-1}x^{-1}x^ax^ay^{h-1}y^{-1}y^ay^a \\ x^{h-1}y^{h-1}y^{-1}x^{-1}x^ay^ax^a &= x^{h-1}x^{-1}x^ax^ay^{h-1}y^{-1}y^a \end{aligned}$$

Since $y^{h-1} \in Z(G)$,

$$y^{-1}x^{-1}x^a y^a x^a = x^{-1}x^a x^a y^{-1} y^a$$

Substituting x by a, we get $y^{-1}y^a a = ay^{-1}y^a$ and so $[y,a]a = a[y,a]$. Thus G is a 2-Engel group. It follows by Lemma 2.4 (iii) that $\exp(G')$ divides $h(h-1)/2$.
On the other hand G is nilpotent and the map $x \mapsto x^h$ is an invertible rational endomorphism of G and $h \notin \{1,-1\}$, by Lemma 2.5 $\exp(G)$ is finite and $gcd(h, exp(G)) = 1$.

Conversely, let $a_1, a_2, \dots, a_r \in G$ and $h_1, h_2, \dots, h_r \in \mathbb{Z}$ such that $h = h_1 + \cdots + h_r$. Consider the map f on G defined by

$$f(x) = (x^{a_1})^{h_1} \dots (x^{a_r})^{h_r} = x^h \prod_{i=1}^{r} [x^{h_i}, a_i][x^{h_i}, a_i, x^{h-(h_1+\cdots+h_i)}].$$

Let $e = exp(G)$ and $e' = exp(G')$. We have $hu + ev = 1$ and $h(h-1) = 2\lambda e'$ for some $u, v, \lambda \in \mathbb{Z}$. Then $h(h-1)u + ev(h-1) = h-1$. Let $x, y \in G$. Therefore, we have

$$[x,y]^{h-1} = [x,y]^{h(h-1)u} = [x,y]^{2\lambda e' u} = 1.$$

It follows that $[x,y]^h = [x,y]$ for all $x, y \in G$. Since G is 2-Engel, the commutator $[x^{h_i}, a_i, x^{h-(h_1+\cdots+h_i)}]$ is trivial. Thus

$$\begin{aligned} f(x)f(y) &= x^h \prod_{i=1}^{r} [x^{h_i}, a_i] \cdot y^h \prod_{i=1}^{r} [y^{h_i}, a_i] \\ &= x^h y^h y^{-h} \prod_{i=1}^{r} [x, a_i]^{h_i} y^h \prod_{i=1}^{r} [y, a_i]^{h_i} = x^h y^h \Big(\prod_{i=1}^{r} [x, a_i]^{h_i} \Big)^{y^h} \prod_{i=1}^{r} [y, a_i]^{h_i} \\ &= x^h y^h \prod_{i=1}^{r} ([x, a_i][x, a_i, y]^h [y, a_i])^{h_i} = x^h y^h \prod_{i=1}^{r} ([x, a_i][x, a_i, y][y, a_i])^{h_i} \end{aligned}$$

On the other hand,

$$f(xy) = (xy)^h \prod_{i=1}^{r} [(xy)^{h_i}, a_i]$$

$$f(xy) = x^h y^h [y,x]^{h(h-1)/2} \prod_{i=1}^{r} [xy, a_i]^{h_i} = x^h y^h \prod_{i=1}^{r} ([x, a_i][x, a_i, y][y, a_i])^{h_i} = f(x)f(y).$$

Hence f is an endomorphism of G. Since $h \notin \{1,-1\}$ and $exp(G)$ is finite and satisfies gcd(h,exp(G)) = 1,then, by Lemma 2.5 $f \in End_r^*(G)$.

(ii) If G satisfies $\mathcal{E}(h)$, it is easy to see that G is 2-Engel with the only condition $exp(G')$ is finite and divides $h(h-1)/2$ by Lemmas 2.3 and 2.4. □

We remark that a group G satisfies $\mathcal{I}(-1)$ if and only if it is abelian

Proof of proposition 1.3. By theorem 1.3 [[1], p. 291-292], the quotient group $\frac{G}{Z_7(G)}$ is $3-$abelian. By lemma 2.3 for $n = 3$, the map $\bar{x} \mapsto \bar{x}^2$ is an endomorphism of $\frac{G}{Z_7(G)}$. then $\frac{G}{Z_7(G)}$ is abelian and G is nilpotent of class at most 8. □

Question 2.6. Let G be a finite group and $\varphi, \psi \in End_r^*(G)$ be such that $\varphi(x) = x^{a_1} \cdots x^{a_n}$ and $\psi(x) = x^{b_1} \cdots x^{b_n}$ for all $x \in G$. If $\varphi = \psi$, then what can be said about $a_1, \ldots, a_n, b_1, \ldots, b_n$?

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Received: July 3, 2007