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On the Annihilators of Power Values of Commutators with Derivations

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Abstract

Let R be a prime ring of char $R \neq 2$, d a nonzero derivation of R, ρ a nonzero right ideal of R and $0 \neq b \in R$ such that $b[[d^2[x, y], [x, y]]^n = 0$ for all $x, y \in \rho$, $n \geq 1$ fixed integer. If $[\rho, \rho]\rho \neq 0$ then either $b\rho = 0$ or $d(\rho)\rho = 0$.

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Throughout this paper, R always denotes a prime ring with extended centroid C and Q its two sided Martindale ring of quotient. By d we mean a nonzero derivation of R. For $x, y \in R$, the commutator of x, y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$.

A well known result proved by Posner [15] states that R must be commutative if $[d(x), x] \in Z(R)$ for all $x \in R$. Lanski [11] generalized this result by proving that if L is a noncommutative Lie ideal of R such that $[d(x), x] \in Z(R)$ for all $x \in L$, then char $R = 2$ and R satisfies S_4, the standard identity in four variables. In [1], Carini and Filippis studied this identity in more general

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situation by considering the power values of commutators with derivation on noncentral Lie ideals. More precisely, they proved that if char $R \neq 2$, L a noncentral Lie ideal of R and $n \geq 1$ a fixed integer such that $[d(x), x]^n \in Z(R)$ for all $x \in L$, then R satisfies S_4. Filippis studied this identity with annihilator condition in [7]. He proved that if char $R \neq 2$, L a noncentral Lie ideal of R and $a \in R$ such that $a[d(x), x]^n \in Z(R)$ for all $x \in L$, $n \geq 1$ a fixed integer, then either $a = 0$ or R satisfies S_4. In a recent paper [6] of Filippis, the following situation is considered: let char $R \neq 2$, d a nonzero derivation of R, I a nonzero right ideal of R and $a \in R$. He proved that if $a[d[x, y], [x, y]] = 0$ for all $x, y \in I$, then either $S_4(I, I, I, I)I = 0$ or $aI = ad(I) = 0$.

In the present paper, our aim is to consider the annihilator condition of power values of commutator with derivation of second order in prime rings. More precisely we will prove the following:

Theorem 2.1 *Let R be a prime ring of char $R \neq 2$, d a nonzero derivation of R, ρ a nonzero right ideal of R and $0 \neq b \in R$ such that $b[[d^2[x, y], [x, y]]^n = 0$ for all $x, y \in \rho$, $n \geq 1$ fixed integer. If $[\rho, \rho]\rho \neq 0$ then either $b\rho = 0$ or $d(\rho)\rho = 0$.*

Example. Let $R = M_k(F)$, $k > 1$, the ring of all $k \times k$ matrices over the field F and $\rho = e_{11}R$. Any derivation $\delta : F \to F$ induces another one in $R = M_k(F)$ as follows: $d : R \to R$ such that $d(\sum_{i,j} \alpha_{ij} e_{ij}) = \sum_{i,j} \delta(\alpha_{ij}) e_{ij}$, where $\alpha_{ij} \in F$ and e_{ij} denotes the usual matrix unit in R. In this case we find that for $b = e_{21} \neq 0$, $b[d^2[x, y], [x, y]]^n = 0$ for all $x, y \in \rho$, for any $n \geq 1$. Clearly, then $[\rho, \rho]\rho = 0$ but $d(\rho)\rho \neq 0$ and $b\rho \neq 0$.

For the sake of completeness we recall some basic notations, definitions and some easy consequences of the result of Kharchenko [9] about the differential identities on a prime ring R. First, we denote by $Der(Q)$ the set of all derivations on Q. By a derivation word Δ of R we mean $\Delta = d_1 d_2 d_3 \ldots d_m$ for some derivations d_i of R. For $x \in R$, we denote by x^Δ the image of x under Δ, that is $x^\Delta = (\cdots(x^{d_1})^{d_2} \cdots)^{d_m}$. By a differential polynomial, we mean a generalized polynomial, with coefficients in Q, of the form $\Phi(x_i^{\Delta_j})$ involving noncommutative indeterminates x_i on which the derivations words Δ_j act as unary operations. $\Phi(x_i^{\Delta_j}) = 0$ is said to be a differential identity on a subset T of Q if it vanishes for any assignment of values from T to its indeterminates x_i.

Now let D_{int} be the C-subspace of $Der(Q)$ consisting of all inner derivations on Q. By Kharchenko's theorem [9, Theorem 2], we have the following result:

Let R be a prime ring of characteristic different from 2. If two nonzero derivations d and δ are C-linearly independent modulo D_{int} and $\Phi(x_i^{\Delta_j})$ is a differential identity on R, where Δ_j are derivations words of the following form $\delta, d, \delta^2, \delta d, d^2$, then $\Phi(y_{ji})$ is a generalized polynomial identity on R, where y_{ji} are distinct indeterminates.

As a particular case, we have:

If d is a nonzero derivation on R and $\Phi(x_1, \ldots, x_n, x_1^d, \ldots, x_n^d, x_1^{d^2}, \ldots, x_n^{d^2})$ is a differential identity on R, then one of the following holds:

(i) either $d \in D_{int}$

or (ii) R satisfies the generalized polynomial identity $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)$

Denote by $Q *_C C\{X, Y\}$ the free product of the C-algebra Q and $C\{X, Y\}$, the free C-algebra in noncommuting indeterminates X, Y.

1 The case for $\rho = R$.

Lemma 1.1 *Let $R = M_2(F)$, the ring of all 2×2 matrices over a field F of characteristic $\neq 2$. If $a, b \in R$ such that $b[[a, [a, [x, y]]], [x, y]]^n = 0$ for all $x, y \in R$, $n \geq 1$ fixed integers, then either $b = 0$ or $a \in F \cdot I_2$.*

Proof. Let $b \neq 0$. We know the fact that for any $x, y \in M_2(F)$, $[x, y]^2 \in Z(M_2(F))$. Choose $x, y \in R$ such that $[[a, [a, [x, y]]], [x, y]]^2 \neq 0$. Now in our assumption $b[[a, [a, [x, y]]], [x, y]]^n = 0$, we may assume that n is even, because if n is not even, we multiply $[[a, [a, [x, y]]], [x, y]]$ from right in both sides to make it even. Now since n is even, $[[a, [a, [x, y]]], [x, y]]^n \in Z(R)$. Since $[[a, [a, [x, y]]], [x, y]]^2 \neq 0$, $[[a, [a, [x, y]]], [x, y]]^n$ is unit. Right multiplying $([[a, [a, [x, y]]], [x, y]]^n)^{-1}$ from right side in our assumption we obtain that $b = 0$, a contradiction. Thus $[[a, [a, [x, y]]], [x, y]]^2 = 0$ for all $x, y \in R$. By assuming $x = e_{11}, y = e_{12}$, we have the identity $0 = [[a, [a, [x, y]]], [x, y]]^2 e_{12} = 2^2 a_{21}^4 e_{12}$, implying that $a_{21} = 0$. Similarly, by assuming $x = e_{22}, y = e_{21}$, we get $0 = [[a, [a, [x, y]]], [x, y]]^2 = 2^2 a_{12}^4 e_{21}$, implying that $a_{12} = 0$. Thus a is diagonal matrix. Again, by assuming $x = e_{11}, y = e_{12} - e_{21}$, we get $0 = [[a, [a, [x, y]]], [x, y]]^2 = -2^2(a_{11} - a_{22})^2(e_{11} + e_{22})$ which gives $a_{11} = a_{22}$. Hence $a \in F \cdot I_2$.

Theorem 1.2 *Let R be a noncommutative prime ring of char $R \neq 2$, d a nonzero derivation of R, $b \in R$ and $n \geq 1$ fixed integers. If $b[d^2[x, y], [x, y]]^n = 0$ for all $x, y \in R$, then $b = 0$.*

Proof. Suppose on the contrary that $b \neq 0$. By assumption we have that $b[[d^2(x), y] + 2[d(x), d(y)] + [x, d^2(y)], [x, y]]^n = 0$ for all $x, y \in R$. If d is not Q-inner then by Kharchenko's theorem [9] we have

$$b[[r_1, y] + 2[r_2, r_3] + [x, r_4], [x, y]]^n = 0$$

for all $x, y, r_1, r_2, r_3, r_4 \in R$. In particular, for $r_1 = r_4 = 0$, we have by using char $R \neq 2$ that

$$b[[r_2, r_3], [x, y]]^n = 0 \tag{1}$$

for all $x, y, r_2, r_3 \in R$. Let $w = [[u,v],[x,y]]^n$. Then $bw = 0$. From (1) we can write $b[[qwpb, q], [w, pb]]^n = 0$ for all $p, q \in R$. Since $bw = 0$, it reduces to $(bqwp)^{2n+1} = 0$ for all $p, q \in R$. By Levitzki's lemma, $bqwp = 0$ for all $p, q \in R$. Since R is prime either $b = 0$ or $w = 0$. Since $b \neq 0$, $w = [[u,v],[x,y]]^n = 0$ for all $x, y, u, v \in R$. This is a polynomial identity and hence there exists a field F such that $R \subseteq M_k(F)$ with $k > 1$ and R and $M_k(F)$ satisfy the same polynomial identity [10, Lemma 1]. But by choosing $u = e_{12}$, $v = e_{11}$, $x = e_{21}$ and $y = e_{11}$, we get

$$0 = [[u,v],[x,y]]^n = (-1)^n\Big(e_{11} + (-)^n e_{22}\Big)$$

which is a contradiction.

Next suppose that d is Q-inner derivation, say $d = ad(a)$ for some $a \in Q$ i.e., $d(x) = [a, x]$ for all $x \in R$, then we have

$$b[[a,[a,[x,y]]],[x,y]]^n = 0$$

for all $x, y \in R$. By Chuang [2, Theorem 2], this GPI is also satisfied by Q i.e.,

$$f(x,y) = b[[a,[a,[x,y]]],[x,y]]^n = 0 \tag{2}$$

for all $x, y \in Q$.

In case the center C of Q is infinite, we have $f(x,y) = 0$ for all $x, y \in Q \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [3, Theorem 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according to C finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed and $f(x,y) = 0$ for all $x, y \in R$.

Now consider the following two cases.

Case I. *R satisfies a nontrivial GPI.*

By Martindale's theorem [14], R is then a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [8, p.75] R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank. If V is finite dimensional over C, then the density of R on V implies that $R \cong M_k(C)$ where $k = \dim_C V$.

Suppose that $\dim_C V \geq 3$.

We show that for any $v \in V$, v and av are linearly C-dependent. Suppose that v and av are linearly independent for some $v \in V$. If $a^2v \notin Span_C\{v, av\}$, then v, av, a^2v are linearly independent over C. By density there exist $x, y \in R$ such that

$$\begin{aligned} xv &= v, \quad xav = 0, \quad xa^2v = av, \\ yv &= 0, \quad yav = v, \quad ya^2v = a^2v. \end{aligned}$$

Then $[x,y]v=0$, $[x,y]av=v$ and $[x,y]a^2v=av-v$. Hence

$$0=b[[a,[a,[x,y]]],[x,y]]^n v=-bv.$$

Again, if $a^2v\in Span_C\{v,av\}$, then $a^2v=\alpha v+\beta av$ for some $\alpha,\beta\in C$. Since v, av are linearly independent over C. By density there exist $x,y\in R$ such that

$$\begin{aligned} xv &= v, \quad xav=0,\\ yv &= 0, \quad yav=v. \end{aligned}$$

Then again

$$0=b[[a,[a,[x,y]]],[x,y]]^n v=-2bv.$$

This implies that if $bv\neq 0$, then v and av are linearly C-dependent. Now choose $v\in V$ such that v and av are linearly C-independent. Set $W=Span_C\{v,av\}$. Then $bv=0$. Since $b\neq 0$, there exists $w\in V$ such that $bw\neq 0$ and then $b(v-w)=bw\neq 0$. By the previous argument we have that w, aw are linearly C-dependent and $(v-w), a(v-w)$ too. Thus there exist $\alpha,\beta\in C$ such that $aw=\alpha w$ and $a(v-w)=\beta(v-w)$. Then $av=\beta(v-w)+aw=\beta(v-w)+\alpha w$ i.e., $(\alpha-\beta)w=av-\beta v\in W$. Now $\alpha=\beta$ implies that $av=\beta v$, a contradiction. Hence $\alpha\neq\beta$ and so $w\in W$. Again, if $u\in V$ with $bu=0$ then $b(w+u)\neq 0$. So, $w+u\in W$ forcing $u\in W$. Thus it is observed that $w\in V$ with $bw\neq 0$ implies $w\in W$ and $u\in V$ with $bu=0$ implies $u\in W$. This implies that $V=W$ i.e., $\dim_C V=2$, a contradiction.

Hence, in any case, v and av are linearly C-dependent for all $v\in V$. Thus for each $v\in V$, $av=\alpha_v v$ for some $\alpha_v\in C$. It is very easy to prove that α_v is independent of the choice of $v\in V$. Thus we can write $av=\alpha v$ for all $v\in V$ and $\alpha\in C$ fixed.

Now let $r\in R$, $v\in V$. Since $av=\alpha v$,

$$[a,r]v=(ar)v-(ra)v=a(rv)-r(av)=\alpha(rv)-r(\alpha v)=0.$$

Thus $[a,r]v=0$ for all $v\in V$ i.e., $[a,r]V=0$. Since $[a,r]$ acts faithfully as a linear transformation on the vector space V, $[a,r]=0$ for all $r\in R$. Therefore $a\in Z(R)$ implies $d=0$, a contradiction.

Thus $\dim_C V=2$. This gives $R\cong M_2(C)$. By Lemma 1.1, $a\in C\cdot I_2$ that is $d=0$, a contradiction.

<u>Case II.</u> *R does not satisfy any nontrivial GPI.*

Since $d\neq 0$, $a\notin C$. Let $T=R*_C C\{x,y\}$, the free product of R and $C\{x,y\}$, the free C-algebra in noncommuting indeterminates x and y. Since R does not satisfy any nontrivial GPI, we have that $b[[a,[a,[x,y]]],[x,y]]^n$ is the zero element in the free product T that is

$$b[a^2[x,y]-2a[x,y]a+[x,y]a^2,[x,y]]^n=0 \tag{3}$$

in T. Since $a \notin C$, a and 1 are linearly C-independent. If $1, a, a^2$ are linearly C-independent, then from (3),

$$b[a^2[x,y] - 2a[x,y]a + [x,y]a^2, [x,y]]^{n-1}(-2[x,y]a[x,y]a) = 0$$

and so $b(-2[x,y]a[x,y]a)^n = 0$, implying that $b = 0$, a contradiction. Again, if $1, a, a^2$ are linearly C-dependent i.e., $a^2 = \alpha + \beta a$ for some $\alpha, \beta \in C$, then from (3),

$$b[a^2[x,y] - 2a[x,y]a + [x,y]a^2, [x,y]]^{n-1}(-2[x,y]a[x,y]a + \beta[x,y]^2 a) = 0$$

that is

$$b[a^2[x,y] - 2a[x,y]a + [x,y]a^2, [x,y]]^{n-1}[x,y](-2a[x,y]a + \beta[x,y]a) = 0.$$

Again, since $a \notin C$,

$$b[a^2[x,y] - 2a[x,y]a + [x,y]a^2, [x,y]]^{n-1}[x,y](-2a[x,y]a) = 0.$$

By the same argument, we have $b(-2[x,y]a[x,y]a)^n = 0$, implying that $b = 0$, a contradiction.

2 The case for one-sided ideal.

In this section we will prove the main theorem of the paper:

Theorem 2.1 *Let R be a prime ring of char $R \neq 2$, d a nonzero derivation of R, ρ a nonzero right ideal of R and $0 \neq b \in R$ such that $b[[d^2[x,y],[x,y]]^n = 0$ for all $x, y \in \rho$, $n \geq 1$ fixed integer. If $[\rho,\rho]\rho \neq 0$ then either $b\rho = 0$ or $d(\rho)\rho = 0$.*

We begin with the following lemma

Lemma 2.2 *Let ρ be a nonzero right ideal of R with char $R \neq 2$ and $b \in R$ such that $b\rho \neq 0$. Suppose that d is a nonzero derivation of R such that $b[d^2[x,y],[x,y]]^n = 0$ for all $x, y \in \rho$ and $n \geq 1$ fixed integer. If $d(\rho)\rho \neq 0$ then R satisfies nontrivial generalized polynomial identity (GPI).*

Proof. Suppose on the contrary that R does not satisfy any nontrivial GPI. We may assume that R is noncommutative, otherwise R satisfies trivially a nontrivial GPI. Now we consider the two cases

Case I. Suppose that d is an Q-inner derivation induced by an element $a \in Q$. Let $y \in \rho$. Since $d(\rho)\rho \neq 0$, by [12, Lemma 3], there exists $x \in \rho$ such that ax and x are linearly C-independent. Let $A(X,Y) = [[a,[a,[xX,yY]]],[xX,yY]]$.

Then by assumption $bA(X,Y)^n$ is a GPI for R and so $bA(X,Y)^n = 0$ in $Q *_C C\{X,Y\}$. Expansion of it yields that

$$bA(X,Y)^{n-1}\Big\{a^2[xX,yY]^2 - 2a[xX,yY]a[xX,yY] + [xX,yY]a^2[xX,yY] \\ -[xX,yY]a^2[xX,yY] + 2[xX,yY]a[xX,yY]a - [xX,yY]^2a^2\Big\} = 0 \quad (4)$$

in $Q *_C C\{X,Y\}$. Now since $d \neq 0$, $a \notin C$ and so $\{1,a\}$ is C-independent. If $a^2 \notin span_C\{1,a\}$, the equation (4) implies that

$$bA(X,Y)^{n-1}(2[xX,yY]a[xX,yY]a) = 0$$

in $Q *_C C\{X,Y\}$. This implies

$$2bA(X,Y)^{n-1}[xX,yY]axXyYa = 0. \quad (5)$$

Again, if $a^2 \in span_C\{1,a\}$, then $a^2 = \alpha + \beta a$ for some $\alpha, \beta \in C$. Now replacing a^2 with $\alpha + \beta a$ in (4), we get by using $\{1,a\}$ C-independent that

$$bA(X,Y)^{n-1}(2[xX,yY]a[xX,yY]a - \beta[xX,yY]^2a) = 0.$$

This can be re-written as

$$bA(X,Y)^{n-1}[xX,yY](2a-\beta)[xX,yY]a = 0$$

in $Q *_C C\{X,Y\}$ which implies that

$$bA(X,Y)^{n-1}[xX,yY](2a-\beta)xXyYa = 0.$$

Since $\{ax, x\}$ is C-independent,

$$2bA(X,Y)^{n-1}[xX,yY]axXyYa = 0 \quad (6)$$

which is same as equation (5). Thus in any case $a^2 \notin span_C\{1,a\}$ or $a^2 \in span_C\{1,a\}$, we have

$$2bA(X,Y)^{n-1}[xX,yY]axXyYa = 0$$

in $Q *_C C\{X,Y\}$ and so

$$2bA(X,Y)^{n-1}xXyYaxXyYa = 0 \quad (7)$$

in $Q *_C C\{X,Y\}$. Similarly, expanding the term $A(X,Y)$ and using the fact $\{x, ax\}$ is C-independent, (7) reduces in both cases $a^2x \notin span_C\{x, ax\}$ and $a^2x \in span_C\{x, ax\}$ to

$$2^n b[xX,yY]axXyYa(xXyYa)^{2(n-1)} = 0$$

and so

$$2^n byYxXaxXyYa(xXyYa)^{2(n-1)} = 0$$

in $Q *_C C\{X, Y\}$, implying that $by = 0$ for all $y \in \rho$. Thus $b\rho = 0$, a contradiction.

Case II. Suppose that d is not Q-inner derivation. Since $b\rho \neq 0$, we choose $x \in \rho$ such that $bx \neq 0$ and $X, Y \in R$.

By our assumption we have that R satisfies

$$b[d^2[xX, xY], [xX, xY]]^n = 0$$

that is

$$\begin{aligned} &b[[d^2(x)X + xd^2(X) + 2d(x)d(X), xY] + 2[d(x)X + xd(X), d(x)Y + xd(Y)] \\ &+[xX, d^2(x)Y + xd^2(Y) + 2d(x)d(Y)], [xX, xY]]^n = 0. \end{aligned}$$

By Kharchenko's theorem [9],

$$\begin{aligned} &b[[d^2(x)X + xs_1 + 2d(x)r_1, xY] + 2[d(x)X + xr_1, d(x)Y + xr_2] \\ &+[xX, d^2(x)Y + xs_2 + 2d(x)r_2], [xX, xY]]^n = 0 \end{aligned}$$

for all $X, Y, r_1, r_2, s_1, s_2 \in R$. In particular, for $r_1 = r_2 = s_2 = 0$, we have

$$\begin{aligned} &b\Big([[d^2(x)X, xY], [xX, xY]] + [[xs_1, xY], [xX, xY]] + 2[[d(x)X, d(x)Y], [xX, xY]] \\ &+[[xX, d^2(x)Y], [xX, xY]]\Big)^n = 0 \end{aligned}$$

for all $X, Y, s_1 \in R$. In the expansion of it, since the term $b[[xs_1, xY], [xX, xY]]^n$ appears nontrivially, $b[[xs_1, xY], [xX, xY]]^n = 0$ which is a nontrivial GPI for R, a contradiction.

We are now ready to prove our main Theorem.

Proof of Theorem 2.1. Suppose that $d(\rho)\rho \neq 0$, $b\rho \neq 0$ and then we derive a contradiction. By Lemma 2.2, R is a prime GPI-ring, so is also Q by [2]. Since Q is centrally closed over C, it follows from [14] that Q is a primitive ring with $H = Soc(Q) \neq 0$.

By our assumption and by [13], we may assume that

$$b[d^2[x, y], [x, y]]^n = 0 \tag{8}$$

is satisfied by ρQ and hence by ρH. Let $e = e^2 \in \rho H$ and $y \in H$. Then replacing x with e and y with $ey(1-e)$ in (8) and then right multiplying it by e we obtain that

$$\begin{aligned} 0 &= b[d^2[e, ey(1-e)], [e, ey(1-e)]]^n e \\ &= b[d^2(ey(1-e)), ey(1-e)]^n e \\ &= b\Big(-ey(1-e)d^2(ey(1-e))\Big)^n e. \end{aligned} \tag{9}$$

Now we have the fact that for any idempotent e, $d(y(1-e))e = -y(1-e)d(e)$ and so

$$\begin{aligned}(1-e)d^2(ey(1-e))e &= (1-e)d\Big(d(e)ey(1-e)+ed(ey(1-e))\Big)e \\ &= (1-e)d\Big(d(e)ey(1-e)\Big)e+(1-e)d\Big(ed(ey(1-e))\Big)e \\ &= -(1-e)d(e)ey(1-e)d(e)+(1-e)d(e)d(ey(1-e))e \\ &= -(1-e)d(e)ey(1-e)d(e)-(1-e)d(e)ey(1-e)d(e) \\ &= -2(1-e)d(e)ey(1-e)d(e).\end{aligned}$$

Using this fact, (9) reduces to

$$\begin{aligned}0 &= (-1)^n b\Big(ey(1-e)(-2d(e)ey(1-e)d(e))\Big)^n e \\ &= (-1)^{2n}2^n b\Big((ey(1-e)d(e))^2\Big)^n e \\ &= 2^n be\Big(y(1-e)d(e)e\Big)^{2n}\end{aligned}$$

for all $y \in H$. Since char $R \neq 2$, we have by [5, Theorem 2] that $bey(1-e)d(e)e = 0$ for all $y \in H$. By primeness of H, $be = 0$ or $(1-e)d(e)e = 0$. By [4, Lemma 1], since H is a regular ring, for each $r \in \rho H$, there exists an idempotent $e \in \rho H$ such that $r = er$ and $e \in rH$. Hence $be = 0$ gives $br = ber = 0$ and $(1-e)d(e)e = 0$ gives $(1-e)d(e) = (1-e)d(e^2) = (1-e)d(e)e = 0$ and so $d(e) = ed(e) \in eH \subseteq \rho H$ and $d(r) = d(er) = d(e)er + ed(er) \in \rho H$. Hence for each $r \in \rho H$, either $br = 0$ or $d(r) \in \rho H$. Thus ρH is the union of its two additive subgroups $\{r \in \rho H | br = 0\}$ and $\{r \in \rho H | d(r) \in \rho H\}$. Hence $b\rho H = 0$ and $d(\rho H) \subseteq \rho H$. The case $b\rho H = 0$ gives $b\rho = 0$, a contradiction. Thus $d(\rho H) \subseteq \rho H$. Set $J = \rho H$. Replacing b with a nonzero element in Jb, we may assume that $b \in J$. Then $\overline{J} = \frac{J}{J\cap\, l_H(J)}$, a prime C-algebra with the derivation $\overline{d}$ such that $\overline{d}(\overline{x}) = \overline{d(x)}$, for all $x \in J$. By assumption we have that

$$\overline{b}[\overline{d}^2[\overline{x},\overline{y}],[\overline{x},\overline{y}]]^n = 0$$

for all $\overline{x},\overline{y} \in \overline{J}$. By Theorem 1.2, we have either $\overline{d} = 0$, $\overline{b} = 0$, $\overline{\rho H}$ is commutative. Therefore we have that either $d(\rho H)\rho H = 0$, $b\rho H = 0$ or $[\rho H, \rho H]\rho H = 0$. Now $d(\rho H)\rho H = 0$ implies $0 = d(\rho\rho H)\rho H = d(\rho)\rho H\rho H$ and so $d(\rho)\rho = 0$. $b\rho H = 0$ implies $b\rho = 0$. $[\rho H, \rho H]\rho H = 0$ implies $0 = [\rho\rho H, \rho H]\rho H = [\rho, \rho H]\rho H\rho H$ and so $[\rho, \rho H]\rho = 0$ and then $0 = [\rho, \rho\rho H]\rho = [\rho, \rho]\rho H\rho$ implying $[\rho, \rho]\rho = 0$. Thus in all the cases we have contradiction. This completes the proof of the theorem.

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