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On the Prime Divisors of Conjugacy Lengths of Solvable Groups

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Abstract

Suppose that G is a finite group. We prove that if $G/F(G)$ is solvable of odd order or supersolvable; and G does not contain abelian normal non-central Sylow subgroups, then $|c\rho(G)| \leq 3c\sigma(G)$. Let m be the total number of abelian Sylow subgroups and n the total number of nonabelian Sylow subgroups of G. If $n \geq 2m-3$, then $|c\rho(G)| \leq 3\sigma(G)$.

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Let G be a group of finite order. Write x^G to denote a conjugacy class of x in G, and $|x^G|$ the conjugacy class length of x in G. By $\pi(G)$, denote the prime divisor set of the order of G. Let $c\rho(G)$ to denote the set of distinct prime divisors dividing the conjugacy class lengths of G and $|c\rho(G)|$ the cardinality of $c\rho(G)$. Let $c\sigma(G)$ to denote the maximum number of differnt prime divisors dividing a single conjugacy class length in G. It was conjectured that $|c\rho(G)| \leq 2\sigma(G)$ for a finite group. It was shown that this is true when $c\sigma(G)$ equals to $1, 2$ or 3. Generally, Casolo proved [2, Theorem 33.10]that this conjecture is true except possibly when G is p-nilpotent with abelian Sylow p-subgroups for at least two prime divisors dividing $|G|$. Furthermore, Casolo and Dolfi disproved this conjecture by a series of group examples which are metabelian and supersolvable. However these examples implies that it seems reasonable that $|c\rho(G)| \leq 3\sigma(G)$ for the finite groups. In this note, we give some evidence for this. Actually we prove that $|c\rho(G)| \leq 3\sigma(G)$ for the solvable groups of odd order or the supersolvable groups, which do not contain abelian normal non-central Sylow subgroups.

Lemma 1. *Assume that G is a solvable group of odd order and V is a faithful and completely reducible G-module. Then there exist $v, w \in V$ such that $C_G(v) \cap C_G(w) = 1$.*

Proof. This is [1, Theorem 3]. ☐

Lemma 2. *Suppose that V is a completely reducible and faithful G-module for a supersolvable group G. Then there exist v and w in V such that $C_G(v) \cap C_G(w) = 1$.*

Proof. This is Theorem A of [4]. ☐

Theorem 3. *Suppose that $G/F(G)$ is solvable of odd order where $F(G)$ denotes Fitting subgroup of G. If G does not posses abelian normal non-central Sylow subgroups, then $|c\rho(G)| \leq 3c\sigma(G)$.*

Proof. By Gaschütz's theorem 1.12 of [3], it follows that $F(G)/\Phi(G))$ is a faithful completely reducible $\overline{G} = G/F(G)$-module. let $V = F(G/\Phi(G))$, we get via lemma 1 above that there are v and w in V such that $C_{\bar{G}}(v) \cap C_{\bar{G}}(w) = 1$. Then $C_{\tilde{G}}(v) \cap C_{\tilde{G}}(w) = F(G)/\Phi(G)$, where $\tilde{G} = G/\Phi(G)$. Also since

$$\begin{aligned} |G/F(G)| &= |\tilde{G} : C_{\tilde{G}}(v) \cap C_{\tilde{G}}(w)| \\ &= |\tilde{G} : C_{\tilde{G}}(v)||C_{\tilde{G}}(v) : C_{\tilde{G}}(v) \cap C_{\tilde{G}(w)}| \\ &= |\tilde{G} : C_{\tilde{G}}(v)||C_{\tilde{G}}(v)C_{\tilde{G}}(w) : C_{\tilde{G}}(w)| \end{aligned}$$

which divides $|\tilde{G} : C_{\tilde{G}}(v)||\tilde{G} : C_{\tilde{G}}(w)|$. Observe that if $v = x\Phi(G)$ and $w = y\Phi(G)$, then $|v^{\tilde{G}}|$ and $|w^{\tilde{G}}|$ divide $|x^G|$ and $|y^G|$ respectively. We have that $\pi(G/F(G)) \subseteq \pi(|x^G|) \cup \pi(|y^G|)$.

Let N be a direct product of all $O_p(G)$ such that $O_p(G)$'s are non-abelian Sylow subgroups of G. It is straightforward to find $z \in N$ with $\pi(|z^N|) = \pi(N)$. Note that $c\rho(G) = \pi(G/Z(G))$. There are not abelian normal non-central Sylow subgroups in G, so that we get that $\pi(G/Z(G)) \subseteq \pi(|x^G|) \cup \pi(|y^G|) \cup \pi(|z^G|)$. Thus we have that $|c\rho(G)| \leq 3c\sigma(G)$, the proof is completed. ☐

Theorem 4. *Suppose that $G/F(G)$ is supersolvable, where $F(G)$ denotes Fitting subgroup of G. If G does not contain normal abelian non-central Sylow subgroups, then $|\rho(G)| \leq 3\sigma(G)$.*

Proof. In fact, this theorem's proof is similar to that of the above theorem, but for the sake of completeness we give a description for its proof. Gashütz's theorem shows that $F(G)/\Phi(G)$ is a completely reducible and faithful $G/F(G)$-module. Lemma 2 implies that there exist $\bar{x}$ and $\bar{y}$ with $C_{\bar{G}}(\bar{x}) \cap C_{\bar{G}}(\bar{y}) = \bar{F}(G)$. Then $\pi(G/F(G)) \subseteq \pi(|x^G|) \cup \pi(|y^G|)$. Since G does not contain abelian normal Sylow non-central subgroups, it follows that we may pick $z \in F(G)$ such that $c\rho(G) \subseteq \pi(|x^G|) \cup \pi(|y^G|) \cup \pi(|z^G|)$. Thus $|c\rho(G)| \leq 3c\sigma(G)$, as desired. ☐

Theorem 5. *Assume that G is a finite group. Let m be the total number of abelian Sylow subgroups and n the total number of nonabelian Sylow subgroups of G. If $n \geq 2m - 3$, then $|c\rho(G)| \leq 3\sigma(G)$.*

Proof. Set $\Delta_p(G) = \{g | g \in G, p \text{ divides} |g^G|\}$. Then $g \notin \Delta_p(G)$ is equivalent with $P \subseteq C_G(g)$ for some $P \in Syl_p(G)$, thus with $g \in C_G(P)$. This shows $G - \Delta_p(G) = \bigcup_{g \in G} C_G(P)^g$ for a fixed $P \in Syl_p(G)$. If we put $m_p = |N_G(P) : C_G(P)|$, then

$$|G - \Delta_p(G)| \leq 1 + |G : N_G(P)|(|C_G(P)| - 1) \leq \frac{|G|}{|N_G(P) : C_G(P)|} = \frac{|G|}{m_p}.$$

This implies that

$$|\Delta_p(G)| \geq |G| - \frac{|G|}{m_p} = \frac{m_p - 1}{m_p}|G|.$$

Let $\Lambda(G) = \{p | m_p = 1\}$. If $m_p = 1$, then $N_G(P) = C_G(P)$, hence P is abelian and G p-nilpotent by Burnside's theorem. If $p \notin \Lambda(G)$, then $|\Delta_p(G)| \geq \frac{m_p - 1}{m_p}|G| \geq \frac{|G|}{2}$. Consider in $c\rho(G) \times (G - \{1\})$ the subset

$$S = \{(p, g) | p \text{ divides} |g^G|\} = \bigcup_{p \in c\rho(G)} (p, \Delta_p(G)).$$

Then

$$\sum_{p \in c\rho(G)} |\Delta_p(G)| = |S| = \sum_{1 \neq g \in G} \pi(|g^G|) \leq (|G| - 1)c\sigma(G).$$

Hence we obtain that

$$(|G| - 1)c\sigma(G) \geq \sum_{p \in c\sigma(G)} \frac{m_p - 1}{m_p}|G|.$$

Therefore

$$c\sigma(G) > \sum_{p \in c\rho(G)} \frac{m_p - 1}{m_p} \geq \frac{n}{2}.$$

Since $n \geq 2m - 3$, it follows that

$$\frac{n+1}{2} \geq \frac{1}{3}(m + n) \geq \frac{1}{3}|c\rho(G)|.$$

Hence $|c\rho(G)| \leq 3c\sigma(G)$, the whole proof is complete. □

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