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Linear Maps of Jacobi-Like Forms Associated to Modular Forms[1]

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Abstract. Jacobi-like forms and modular series are formal power series, whose coefficients are holomorphic functions on the Poincaré upper half plane, satisfying certain transformation formulas under the operation of a discrete subgroup Γ of $SL(2, \mathbb{R})$, and there is a natural isomorphism between the spaces of these two types of series. Given a modular form h for Γ, multiplication by h determines a linear map $\mathcal{L}_h^J$ of Jacobi-like forms and a linear map $\mathcal{L}_h^M$ of modular series, which are not compatible with respect to the above isomorphism. We construct a linear map of modular series compatible with $\mathcal{L}_h^J$ as well as a linear map of Jacobi-like forms compatible with $\mathcal{L}_h^M$.

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1. Introduction

Jacobi-like forms are formal power series, whose coefficients are holomorphic functions on the Poincaré upper half plane, satisfying a certain transformation formula under the operation of a discrete subgroup Γ of $SL(2, \mathbb{R})$. This formula is essentially one of the two equations that must be satisfied by Jacobi forms (cf. [2]). Given a Jacobi-like form $\Phi(z, X)$, its transformation formula determines an expression of each of its coefficients in terms of derivatives of some modular forms for Γ, and each of these modular forms can also be expressed as a linear combination of derivatives of some of the coefficients of $\Phi(z, X)$. These expressions can be used to derive a one-to-one correspondence between Jacobi-like forms and certain sequences of modular forms (see [1, 4]).

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Modular series are formal power series which may be regarded as special types of Jacobi-like forms. In fact, they can be defined by slightly modifying the transformation formula for Jacobi-like forms. Given a sequence of modular forms corresponding to a Jacobi-like form, a modular series can be obtained by using the terms of the sequence as the coefficients of the formal power series. Then the correspondence between Jacobi-like forms and sequences of modular forms can be interpreted as an isomorphism between the space of Jacobi-like forms and the space of modular series.

As in the case of modular forms, weights can be introduced for both Jacobi-like forms and modular series. Given a modular form h for Γ of weight w and a Jacobi-like form $\Phi(z,X)$ of weight λ, by multiplying the coefficients of $\Phi(z,X)$ by h we obtain a Jacobi-like form of weight $\lambda+w$. Thus h determines a linear map $\mathcal{L}^J_h : \mathcal{J}_\lambda(\Gamma) \to \mathcal{J}_{\lambda+w}(\Gamma)$, where $\mathcal{J}_\lambda(\Gamma)$ is the space of Jacobi-like forms of weight λ. Similarly, if $\mathcal{M}_\lambda(\Gamma)$ denotes the space of modular series of weight λ, the same modular form h determines a linear map $\mathcal{L}^M_h : \mathcal{M}_\lambda(\Gamma) \to \mathcal{M}_{\lambda+w}(\Gamma)$ of modular series which sends a modular series $F(z,X) \in \mathcal{M}_\lambda(\Gamma)$ to $h(z)F(z,X) \in \mathcal{M}_{\lambda+w}(\Gamma)$. On the other hand, as was mentioned above, there are isomorphisms $\Xi_\lambda : \mathcal{J}_\lambda(\Gamma) \to \mathcal{M}_\lambda(\Gamma)$ and $\Xi_{\lambda+w} : \mathcal{J}_{\lambda+w}(\Gamma) \to \mathcal{M}_{\lambda+w}(\Gamma)$, but the maps $\mathcal{L}^J_h$ and $\mathcal{L}^M_h$ are not compatible with respect to these isomorphisms. In this paper we construct a linear map of modular series compatible with $\mathcal{L}^J_h$ as well as a linear map of Jacobi-like forms compatible with $\mathcal{L}^M_h$.

2. JACOBI-LIKE FORMS

In this section we review the correspondence between Jacobi-like forms and sequences of modular forms. We interpret this correspondence as an isomorphism between the spaces of Jacobi-like forms and modular series.

Let $\mathcal{H}$ be the Poincaré upper half plane, and let R be the set of all holomorphic functions on $\mathcal{H}$. Then the group $SL(2,\mathbb{R})$ acts on $\mathcal{H}$ as usual by linear fractional transformations, so that we may write

$$\gamma z = \frac{az+b}{cz+d}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. For the same z and γ, we set

$$\mathfrak{J}(\gamma,z) = cz+d, \quad \mathfrak{K}(\gamma,z) = c\mathfrak{J}(\gamma,z)^{-1} = \frac{c}{cz+d}.$$

If $f \in R$, $\gamma \in SL(2,\mathbb{R})$ and $w \in \mathbb{Z}$, we denote by $f \mid_w \gamma$ the element of R defined by

$$(f \mid_w \gamma)(z) = \mathfrak{J}(\gamma,z)^{-w} f(z)$$

for all $z \in \mathcal{H}$. Throughout this paper we fix a discrete subgroup Γ of $SL(2,\mathbb{R})$.

Definition 2.1. Given an integer w, a *modular form for* Γ *of weight* w is an element $f \in R$ satisfying

$$f \mid_w \gamma = f$$

for all $\gamma \in \Gamma$. We denote by $M_w(\Gamma)$ the space of all modular forms of weight w for Γ.

Remark 2.2. We have modified the usual definition of modular forms by suppressing the finiteness condition at the cusps. This allows us, for example, to consider modular forms of negative weight.

Let $R[[X]]$ denote the complex algebra of formal power series in X with coefficients in R. Given a formal power series $\Phi(z, X) \in R[[X]]$, an integer λ, and an element $\gamma \in SL(2, \mathbb{R})$, we set

$$(\Phi \mid^J_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{K}(\gamma,z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X), \tag{2.1}$$

$$(\Phi \mid^M_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X) \tag{2.2}$$

for all $z \in \mathcal{H}$. If γ' is another element of Γ, it can be shown that

$$\Phi \mid^J_\lambda (\gamma\gamma') = (\Phi \mid^J_\lambda \gamma) \mid^J_\lambda \gamma', \quad \Phi \mid^M_\lambda (\gamma\gamma') = (\Phi \mid^M_\lambda \gamma) \mid^M_\lambda \gamma';$$

hence the operations $\mid^J_\lambda$ and $\mid^M_\lambda$ determine right actions of $SL(2, \mathbb{R})$ on $R[[X]]$.

Definition 2.3. Let $\Phi(z, X)$ be a formal power series in $R[[X]]$, and let λ be an integer.

(i) $\Phi(z, X)$ is a *Jacobi-like form for* Γ *of weight* λ if it satisfies

$$(\Phi \mid^J_\lambda \gamma)(z, X) = \Phi(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. We denote by $\mathcal{J}_\lambda(\Gamma)$ the space of all Jacobi-like forms for Γ of weight λ

(ii) $\Phi(z, X)$ is a *modular series for* Γ *of weight* λ if it satisfies

$$(\Phi \mid^M_\lambda \gamma)(z, X) = \Phi(z, X) \tag{2.3}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. We denote by $\mathcal{M}_\lambda(\Gamma)$ the space of all modular series for Γ of weight λ.

Remark 2.4. Given a complex number μ, we may also consider Jacobi-like forms of index μ by using the operation

$$(\Phi \mid^J_{\lambda,\mu} \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mu\mathfrak{K}(\gamma,z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X)$$

for $\gamma \in \Gamma$ instead of (2.2). Then the Jacobi-like forms of index 1 are simply the Jacobi-like forms in Definition 2.3. On the other hand, modular series for Γ of weight λ can be regarded as Jacobi-like forms of weight λ and index 0.

Example 2.5. Let $\Phi_1(z, X)$ and $\Phi_2(z, X)$ be Jacobi-like forms for Γ of weights λ_1 and λ_2, respectively, and set

$$\Phi(z, X) = \Phi_1(z, X)\Phi_2(z, -X).$$

Then from (2.1), (2.2) and Definition 2.3 we obtain

$$\begin{aligned}\Phi(z,X) &= (\Phi_1 \mid^J_\lambda \gamma)(z,X)(\Phi_2 \mid^J_\lambda \gamma)(z,-X)\\ &= \mathfrak{J}(\gamma,z)^{-\lambda_1} e^{-\mathfrak{K}(\gamma,z)X}\Phi_1(\gamma z, \mathfrak{J}(\gamma,z)^{-2}X)\\ &\qquad\qquad \times \mathfrak{J}(\gamma,z)^{-\lambda_2} e^{\mathfrak{K}(\gamma,z)X}\Phi_2(\gamma z, -\mathfrak{J}(\gamma,z)^{-2}X)\\ &= \mathfrak{J}(\gamma,z)^{-\lambda_1-\lambda_2}\Phi(\gamma z, \mathfrak{J}(\gamma,z)^{-2}X)\\ &= (\Phi \mid^M_{\lambda_1+\lambda_2} \gamma)(z,X)\end{aligned}$$

for all $\gamma \in \Gamma$; hence it follows that $\Phi(z,X)$ is a modular series belonging to $\mathcal{M}_{\lambda_1+\lambda_2}(\Gamma)$.

If δ is a nonnegative integer, we set

$$R[[X]]_\delta = X^\delta R[[X]],$$

$$\mathcal{J}_\lambda(\Gamma)_\delta = \mathcal{J}_\lambda(\Gamma) \cap R[[X]]_\delta, \quad \mathcal{M}_\lambda(\Gamma)_\delta = \mathcal{M}_\lambda(\Gamma) \cap R[[X]]_\delta,$$

so that each element of $\mathcal{J}_\lambda(\Gamma)_\delta$ and $\mathcal{M}_\lambda(\Gamma)_\delta$ can be written in the form

$$\Phi(z,X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta}, \quad F(z,X) = \sum_{k=0}^\infty f_k(z) X^{k+\delta},$$

respectively, with $\phi_k, f_k \in R$ for each $k \geq 0$. In this case the condition (2.3) implies that the coefficient f_k of the modular series $F(z,X) \in \mathcal{M}_\lambda(\Gamma)_\delta$ is a modular form belonging to $M_{2k+\lambda}(\Gamma)$ for each $k \geq 0$.

Proposition 2.6. *Let λ and δ be integers with $\delta \geq 0$, and let*

$$\Phi(z,X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta} \in R[[X]]_\delta.$$

Then the following conditions are equivalent:

(i) *The formal power series $\Phi(z,X)$ is a Jacobi-like form belonging to $\mathcal{J}_\lambda(\Gamma)_\delta$.*

(ii) *The coefficients of $\Phi(z,X)$ satisfy*

$$(\phi_k \mid_{2k+2\delta+\lambda} \gamma)(z) = \sum_{r=0}^k \frac{1}{r!}\mathfrak{K}(\gamma,z)^r \phi_{k-r}(z) \tag{2.4}$$

for all $k \geq 0$, $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

(iii) *The coefficients of $\Phi(z,X)$ can be written in the form*

$$\phi_k = \sum_{r=0}^k \frac{1}{r!(2k+2\delta+\lambda-r-1)!} f^{(r)}_{k+\delta-r} \tag{2.5}$$

for all $k \geq 0$, where f_w is a modular form belonging to $M_{2w+\lambda}(\Gamma)$ for each $w \geq \delta$.

Proof. This can be proved by modifying the proof of Proposition 2 in [1], where the case of $\delta = 1$ and $\lambda = 0$ was considered. □

Remark 2.7. Given nonnegative integers k and s, a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is a *quasimodular form for* Γ *of weight* k *and depth* s if there are holomorphic functions $f_0, \dots, f_m$ on $\mathcal{H}$ such that

$$(f \mid_k \gamma)(z) = \sum_{m=0}^{s} f_m(z) \mathfrak{K}(\gamma, z)^m$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$ (see e.g. [3]). The relation (2.4) shows that, for each $k \geq 0$, ϕ_k is a quasimodular form for Γ of weight $2k + 2\delta + \lambda$ and depth k.

Lemma 2.8. *The system of relations* (2.5) *between the coefficients of the Jacobi-like form* $\Phi(z, X)$ *and the corresponding modular forms can be written in the form*

$$f_k = (2k + \lambda - 1) \sum_{r=0}^{k-\delta} (-1)^r \frac{(2k + \lambda - r - 2)!}{r!} \phi^{(r)}_{k-\delta-r} \in M_{2k+\lambda}(\Gamma)$$

for all $k \geq \delta$.

Proof. This also extends one of the results in Proposition 2 in [1] and can be proved in a similar manner. ☐

From Proposition 2.6 and Lemma 2.8 we see that there is a $\mathbb{C}$-linear isomorphism between $\mathcal{J}_\lambda(\Gamma)_\delta$ and the complex linear space of sequences $\{f_\ell\}_{\ell=\delta}^{\infty}$ of modular forms such that $f_\ell \in M_{2\ell+\lambda}(\Gamma)$ for each $\ell \geq \delta$. Thus by setting

$$\Xi_\lambda(\Phi(z, X)) = \sum_{\ell=\delta}^{\infty} f_\ell(z) X^\ell = \sum_{\ell=0}^{\infty} f_{\ell+\delta}(z) X^{\ell+\delta}$$

we obtain the isomorphism

$$\Xi_\lambda : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathcal{M}_\lambda(\Gamma)_\delta \tag{2.6}$$

between Jacobi-like forms and modular series.

Example 2.9. Let f be a modular form belonging to $M_{2\delta+\lambda}(\Gamma)$. Then we see easily that $F_f(z, X) = f(z) X^\delta$ is a modular series belonging to $\mathcal{M}_\lambda(\Gamma)$, and the corresponding sequence $\{f_j\}_{j=\delta}^{\infty}$ of modular forms for Γ with $f_j \in M_{2j+\lambda}(\Gamma)$ satisfies

$$f_j = \begin{cases} f & \text{if } j = \delta; \\ 0 & \text{if } j \neq \delta. \end{cases}$$

Thus, if we write

$$\Xi_\lambda^{-1}(F_f(z, X)) = \sum_{k=0}^{\infty} \phi_{f,k}(z) X^{k+\delta},$$

then from (2.4) we obtain

$$\phi_{f,k} = \sum_{r=0}^{k} \frac{f^{(r)}_{k+\delta-r}}{r!(2k + \lambda + 2\delta - r - 1)!} = \frac{f^{(k)}}{k!(k + \lambda + 2\delta - 1)!}$$

for each $k \geq 0$; hence it follows that

$$\Xi_\lambda^{-1}(F_f(z,X)) = \sum_{k=0}^\infty \frac{f^{(k)}(z)}{k!(k+\lambda+2\delta-1)!} X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta.$$

In particular, the map $f \mapsto \Xi_\lambda^{-1}(F_f(z,X))$ may be regarded as a lifting of modular forms to Jacobi-like forms.

3. LINEAR MAPS OF JACOBI-LIKE FORMS

We fix an integer w and a modular form $h : \mathcal{H} \to \mathbb{C}$ belonging to $M_w(\Gamma)$. If $\Phi(z,X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta}$ is a formal power series in $R[[X]]_\delta$, we set

$$\mathcal{L}_h(\Phi(z,X)) = \sum_{k=0}^\infty h(z)\phi_k(z) X^{k+\delta}.$$

Then we see easily that

$$\mathcal{L}_h(\mathcal{J}_\lambda(\Gamma))_\delta \subset \mathcal{J}_{\lambda+w}(\Gamma)_\delta, \quad \mathcal{L}_h(\mathcal{M}_\lambda(\Gamma)_\delta) \subset \mathcal{M}_{\lambda+w}(\Gamma)_\delta.$$

Thus $\mathcal{L}_h$ induces, by restriction, the linear maps

$$\mathcal{L}_h^J : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathcal{J}_{\lambda+w}(\Gamma)_\delta, \quad \mathcal{L}_h^M : \mathcal{M}_\lambda(\Gamma)_\delta \to \mathcal{M}_{\lambda+w}(\Gamma)_\delta, \tag{3.1}$$

which are not compatible with respect to the isomorphisms Ξ_λ and $\Xi_{\lambda+w}$. In this section we construct a linear map of Jacobi-like forms that is compatible with $\mathcal{L}_h^M$.

As a candidate for the desired linear map of Jacobi-like forms, we consider a linear map

$$\widetilde{\mathcal{L}}_h^J : \mathcal{J}_\lambda(\Gamma)_\delta \to R[[X]]_\delta \tag{3.2}$$

defined by

$$\widetilde{\mathcal{L}}_h^J(\Phi(z,X)) = \sum_{k=0}^\infty \sum_{s=0}^k \sum_{r=0}^{k-s} \sum_{j=0}^s \frac{(-1)^r(2k+2\delta-2s+\lambda-1)}{j!r!(s-j)!} \times \frac{(2k+2\delta+\lambda-2s-r-2)!}{(2k+2\delta+\lambda+w-s-1)!} h^{(j)}(z)\phi_{k-r-s}^{(r+s-j)}(z) X^{k+\delta} \tag{3.3}$$

for $\Phi(z,X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta$.

Theorem 3.1. *The image of the linear map $\widetilde{\mathcal{L}}_h^J$ in (3.2) is contained in $\mathcal{J}_{\lambda+w}(\Gamma)_\delta$. Furthermore, if $\widetilde{\mathcal{L}}_h^J : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathcal{J}_{\lambda+w}(\Gamma)_\delta$ is the same linear map whose codomain is restricted to $\mathcal{J}_{\lambda+w}(\Gamma)_\delta$, then the diagram*

$$\begin{tikzcd} \mathcal{J}_\lambda(\Gamma)_\delta \arrow[r, "\widetilde{\mathcal{L}}_h^J"] \arrow[d, "\Xi_\lambda"'] & \mathcal{J}_{\lambda+w}(\Gamma)_\delta \arrow[d, "\Xi_{\lambda+w}"] \\ \mathcal{M}_\lambda(\Gamma)_\delta \arrow[r, "\mathcal{L}_h^M"] & \mathcal{M}_{\lambda+w}(\Gamma)_\delta \end{tikzcd}$$

is commutative, where Ξ_λ, $\Xi_{\lambda+w}$ and $\mathcal{L}_h^M$ are as in (2.6) and (3.1).

Proof. Given a Jacobi-like form $\Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta$, we set

$$\Xi_\lambda(\Phi(z, X)) = \sum_{\ell=\delta}^{\infty} f_\ell(z) X^\ell \in \mathcal{M}_\lambda(\Gamma).$$

Then from Lemma 2.8 we obtain

$$f_k = (2k + \lambda - 1) \sum_{r=0}^{k-\delta} (-1)^r \frac{(2k + \lambda - r - 2)!}{r!} \phi_{k-\delta-r}^{(r)} \in M_{2k+\lambda}(\Gamma)$$

for each $k \geq \delta$. Thus, if we write

$$(\mathcal{L}_h^M \circ \Xi_\lambda)(\Phi(z, X)) = \sum_{\ell=\delta}^{\infty} \widehat{f}_\ell(z) X^\ell \in \mathcal{M}_{\lambda+w}(\Gamma)_\delta,$$

we see that

$$\widehat{f}_k = h f_k = (2k + \lambda - 1) \sum_{r=0}^{k-\delta} (-1)^r \frac{(2k + \lambda - r - 2)!}{r!} h \phi_{k-\delta-r}^{(r)} \in M_{2k+\lambda+w}(\Gamma)$$

for each $k \geq \delta$. We now assume that

$$(\Xi_{\lambda+w}^{-1} \circ \mathcal{L}_h^M \circ \Xi_\lambda)(\Phi(z, X)) = \sum_{k=0}^{\infty} \widehat{\phi}(z) X^{k+\delta} \in \mathcal{J}_{\lambda+w}(\Gamma)_\delta.$$

Then, using (2.5), we have

$$\begin{aligned}
\widehat{\phi}_k &= \sum_{s=0}^{k} \frac{1}{s!(2k + 2\delta + \lambda + w - s - 1)!} (h f_{k+\delta-s})^{(s)} \\
&= \sum_{s=0}^{k} \frac{2k + 2\delta - 2s + \lambda - 1}{s!(2k + 2\delta + \lambda + w - s - 1)!} \\
&\qquad \times \sum_{r=0}^{k-s} (-1)^r \frac{(2k + 2\delta - 2s + \lambda - r - 2)!}{r!} (h \phi_{k-r-s}^{(r)})^{(s)} \\
&= \sum_{s=0}^{k} \sum_{r=0}^{k-s} \sum_{j=0}^{s} \frac{2k + 2\delta - 2s + \lambda - 1}{s!(2k + 2\delta + \lambda + w - s - 1)!} \\
&\qquad \times (-1)^r \binom{s}{j} \frac{(2k + 2\delta - 2s + \lambda - r - 2)!}{r!} h^{(j)} \phi_{k-r-s}^{(r+s-j)}
\end{aligned}$$

for each $k \geq 0$. From this and (3.3) it follows that

$$(\Xi_{\lambda+w}^{-1} \circ \mathcal{L}_h^M \circ \Xi_\lambda)(\Phi(z, X)) = \widetilde{\mathcal{L}}_h^J(\Phi(z, X)).$$

In particular, we see that

$$\widetilde{\mathcal{L}}_h^J(\Phi(z, X)) \in \mathcal{J}_{\lambda+w}(\Gamma)_\delta$$

and $\mathcal{L}_h^M \circ \Xi_\lambda = \Xi_{\lambda+w} \circ \widetilde{\mathcal{L}}_h^J$; hence the theorem follows. □

Example 3.2. If f is a modular form belonging to $M_{2\delta+\lambda}(\Gamma)$, by Example 2.9 the series

$$\Phi_f(z,X) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!(k+\lambda+2\delta-1)!} X^{k+\delta}$$

is a Jacobi-like form belonging to $\mathcal{J}_\lambda(\Gamma)_\delta$ such that

$$\Xi_\lambda(\Phi_f(z,X)) = f(z)X^\delta \in \mathcal{M}_\lambda(\Gamma).$$

Thus we have

$$\mathcal{L}_h^M(\Phi_f(z,X)) = h(z)f(z)X^\delta \in \mathcal{M}_{\lambda+w}(\Gamma),$$

and, as in Example 2.9, we see that

$$\begin{aligned}\widetilde{\mathcal{L}}_h^J(\Phi_f(z,X)) &= (\Xi_{\lambda+w}^{-1} \circ \mathcal{L}_h^M \circ \Xi_\lambda)(\Phi_f(z,X)) \\ &= \sum_{k=0}^{\infty} \frac{(hf)^{(k)}(z)}{k!(k+\lambda+w+2\delta-1)!} X^{k+\delta},\end{aligned}$$

which is a Jacobi-like form belonging to $\mathcal{J}_\lambda(\Gamma)_{\delta+w}$. In particular, by using $\lambda = -\delta$ we obtain a bilinear map

$$M_w(\Gamma) \times M_\delta(\Gamma) \to \mathcal{J}_{-\delta}(\Gamma)_{\delta+w}$$

sending (h,f) to $\widetilde{\mathcal{L}}_h^J(\Phi_f(z,X))$.

4. LINEAR MAPS OF MODULAR SERIES

Let $\mathcal{L}_h^J : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathcal{J}_{\lambda+w}(\Gamma)_\delta$ with $h \in M_w(\Gamma)$ be as in (3.1), and let $F(z,X)$ be a modular series for Γ belonging to $\mathcal{M}_\lambda(\Gamma)_\delta$ of the form

$$F(z,X) = \sum_{k=\delta}^{\infty} f_k(z)X^k. \tag{4.1}$$

In this section we find a linear map of modular series compatible with $\mathcal{L}_h^J$.

We first define the function $\widehat{f}_k : \mathcal{H} \to \mathbb{C}$ by

$$\begin{aligned}\widehat{f}_k = \sum_{r=0}^{k-\delta} \sum_{j=0}^{r} \sum_{s=0}^{k-\delta-r} & \frac{(-1)^r(2k+\lambda+w-1)}{j!s!(r-j)!} \\ & \times \frac{(2k+\lambda+w-r-2)!}{(2k+\lambda-2r-s-1)!} h^{(j)} f_{k-r-s}^{(r+s-j)}\end{aligned} \tag{4.2}$$

for each $k \geq \delta$ and set

$$\widetilde{\mathcal{L}}_h^M(F(z,X)) = \sum_{k=\delta}^{\infty} \widehat{f}_k(z)X^k,$$

so that we obtain the linear map

$$\widetilde{\mathcal{L}}_h^M : \mathcal{M}_\lambda(\Gamma)_\delta \to R[[X]]_\delta \tag{4.3}$$

defined on the space of modular forms.

Theorem 4.1. *The image of the map* $\widetilde{\mathcal{L}}_h^M$ *in* (4.3) *is contained in* $\mathcal{M}_{\lambda+w}(\Gamma)_\delta$. *Furthermore, if the codomain of* $\widetilde{\mathcal{L}}_h^M$ *is restricted to* $\mathcal{M}_{\lambda+w}(\Gamma)_\delta$, *the diagram*

$$\begin{tikzcd}
\mathcal{J}_\lambda(\Gamma)_\delta \arrow[r, "\mathcal{L}_h^J"] \arrow[d, "\Xi_\lambda"'] & \mathcal{J}_{\lambda+w}(\Gamma)_\delta \arrow[d, "\Xi_{\lambda+w}"] \\
\mathcal{M}_\lambda(\Gamma)_\delta \arrow[r, "\widetilde{\mathcal{L}}_h^M"] & \mathcal{M}_{\lambda+w}(\Gamma)_\delta
\end{tikzcd}$$

is commutative, where Ξ_λ, $\Xi_{\lambda+w}$ *and* $\mathcal{L}_h^J$ *are as in* (2.6) *and* (3.1).

Proof. Given $F(z,X) \in \mathcal{M}_\lambda(\Gamma)_\delta$ as in (4.1), if we write

$$\Xi_\lambda^{-1}(F(z,X)) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta,$$

then (2.5) implies that

$$\phi_k = \sum_{s=0}^{k} \frac{1}{s!(2k+2\delta+\lambda-s-1)!} f_{k+\delta-s}^{(s)}$$

for each $k \geq 0$. Then we see that

$$\mathcal{L}_h^J(\Xi_\lambda^{-1}(F(z,X))) = \sum_{k=0}^{\infty} h(z)\phi_k(z) X^{k+\delta}$$

is a Jacobi-like form belonging to $\mathcal{J}_{\lambda+w}(\Gamma)_\delta$. Applying $\Xi_{\lambda+w}$ to this, we have

$$(\Xi_{\lambda+w} \circ \mathcal{L}_h^J \circ \Xi_\lambda^{-1})(F(z,X)) = \sum_{k=\delta}^{\infty} \widetilde{f}_k(z) X^k,$$

where the coefficients are given by

$$\begin{aligned}
\widetilde{f}_k &= (2k+\lambda+w-1) \sum_{r=0}^{k-\delta} (-1)^r \frac{(2k+\lambda+w-r-2)!}{r!} (h\phi_{k-\delta-r})^{(r)} \\
&= (2k+\lambda+w-1) \sum_{r=0}^{k-\delta} \sum_{j=0}^{r} \sum_{s=0}^{k-\delta-r} (-1)^r \binom{r}{j} \frac{(2k+\lambda+w-r-2)!}{r!} h^{(j)} \\
&\qquad\qquad \times \frac{1}{s!(2k-2r+\lambda-s-1)!} f_{k-r-s}^{(r+s-j)}
\end{aligned}$$

for all $k \geq \delta$. Comparing this with $\widehat{f}_k$ in (4.2), we obtain

$$(\Xi_{\lambda+w} \circ \mathcal{L}_h^J \circ \Xi_\lambda^{-1})(F(z,X)) = \widetilde{\mathcal{L}}_h^M(F(z,X)).$$

In particular, we see that

$$\widetilde{\mathcal{L}}_h^M(F(z,X)) \in \mathcal{M}_{\lambda+w}(\Gamma)_\delta$$

and $\Xi_{\lambda+w} \circ \mathcal{L}_\lambda^J = \widetilde{\mathcal{L}}_h^M \circ \Xi_\lambda$, which prove the theorem. ☐

Example 4.2. Let $F_f(z,X) = f(z)X^\delta$ be a modular series belonging to $\mathcal{M}_\lambda(\Gamma)$ with $f \in M_{2\delta+\lambda}(\Gamma)$. Then as in Example 2.9 we see that

$$\Xi_\lambda^{-1}(F_f(z,X)) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!(k+\lambda+2\delta-1)!} X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta;$$

hence we have

$$\mathcal{L}_h^J(\Xi_\lambda^{-1}(F_f(z,X))) = \sum_{k=0}^{\infty} \frac{h(z)f^{(k)}(z)}{k!(k+\lambda+2\delta-1)!} X^{k+\delta} \in \mathcal{J}_{\lambda+w}(\Gamma)_\delta.$$

Thus we obtain

$$\widetilde{\mathcal{L}}_h^M(F_f(z,X)) = (\Xi_{\lambda+w} \circ \mathcal{L}_\lambda^J \circ \Xi_\lambda^{-1})(F_f(z,X)) = \sum_{k=\delta}^{\infty} f_k(z)X^k,$$

where

$$f_k = (2k+\lambda+w-1) \sum_{r=0}^{k-\delta} (-1)^r \frac{(2k+\lambda+w-r-2)!(hf^{(k-\delta-r)})^{(r)}}{r!(k-\delta-r)!(k+\lambda+\delta-r-1)!}$$

for each $k \geq \delta$.

5. CONCLUDING REMARKS

If R is as in Section 2, a pseudodifferential operator over R is a formal Laurent series in the formal inverse ∂^{-1} of ∂ with coefficients in R, that is, an expression of the form $\sum_{k=-\infty}^{u} h_k(z)\partial^k$ with $u \in \mathbb{Z}$ and $h_k \in R$. Then the linear fractional transformation on $\mathcal{H}$ and the transformation $\partial \mapsto \mathfrak{J}(\gamma,z)^2\partial$ for $\gamma \in \Gamma$ determine an action of Γ on the space of pseudodifferential operator over R, and the invariant elements under this action are known as *automorphic pseudodifferential operators*. As was discussed by Cohen, Manin and Zagier in [1], there is a one-to-one correspondence between Jacobi-like forms and automorphic pseudodifferential operators. By using this correspondence the results in Theorem 3.1 and Theorem 4.1 can easily be interpreted in terms of automorphic pseudodifferential operators.

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