International Journal of Algebra, Vol. 2, 2008, no. 4, 153 - 161

Indecomposable Modules for Domestic Canonical Algebras in Arbitrary Characteristic

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Abstract. We describe explicitly all indecomposable modules of rank 6 over a domestic canonical algebra of quiver type over a field k of arbitrary characteristic. Together with the results given in [5] this yields an explicit description of all preprojective and preinjective indecomposable modules (and of all indecomposable modules if k is algebraically closed) for a domestic canonical algebra of quiver type. In particular for those algebras each preprojective and each preinjective indecomposable module can be represented by matrices whose coefficients are 0 and 1.

Mathematics Subject Classification: 16G20

Keywords: Canonical algebra, domestic type, exceptional module, preprojective module, zero-one matrix

1. Introduction

Let k be a field and $A = kQ/I$ a finite dimensional k-algebra of quiver type. In order to describe finite dimensional left modules over A one has to choose for each vertex of Q a finite dimensional vector space and for each arrow of Q a linear map such that the relations defined by the ideal I are satisfied. Whereas often one has knowledge about the dimension vectors of the indecomposable modules in general, little is known about the linear maps.

In some cases however the problem of an explicit description of the indecomposable modules is solved. Already in 1890 Kronecker [3] classified pairs of $n \times m$-matrices under the natural action of the group $\mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$, in other words finite dimensional modules over the algebra which is called today the Kronecker algebra. In 1972 Gabriel computed all indecomposable representations for Dynkin quivers [1] which was one of the most important steps at the beginning of modern representation theory. Nazarova [6] has described

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[1] The second author was supported by the Polish Scientific Grant KBN 1 P03A 007 27.

the indecomposable modules for the 4-subspace problem, i.e. representations of the quiver $\widetilde{D}_4$ with subspace orientation, however an explicit description of all indecomposable modules for the extended Dynkin quivers is not so easy (compare [4]).

In this paper we study indecomposable modules for domestic canonical algebras of quiver type over a field k. This class of algebras was introduced by Ringel [7] and is related to the extended Dynkin quivers by tilting theory. The problem of an explicit description of the preprojective and preinjective modules over domestic canonical algebras was solved in [5] under the assumption that the characteristic of the field is different from 2. Unfortunately in characteristic 2 some of the modules given there are decomposable. In this paper we complete the classification for arbitrary characteristic.

Recall that a domestic canonical algebra of quiver type Λ is isomorphic to the path algebra of the quiver

$$\begin{tikzcd}
& i_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{p-1}"] & i_{p-1} \arrow[dr, "\alpha_p"] & \\
0 \arrow[ur, "\alpha_1"] \arrow[r, "\beta_1"] \arrow[dr, "\gamma_1"'] & j_1 \arrow[r, "\beta_2"] & \cdots \arrow[r, "\beta_{q-1}"] & j_{q-1} \arrow[r, "\beta_q"] & \infty \\
& k_1 \arrow[r, "\gamma_2"] & \cdots \arrow[r, "\gamma_{s-1}"] & k_{s-1} \arrow[ur, "\gamma_s"'] &
\end{tikzcd}$$

modulo the relation $\gamma_s \dots \gamma_1 = \alpha_p \dots \alpha_1 + \beta_q \dots \beta_1$, where p, q, s is the length of the upper (middle, lower, respectively) arm, and where moreover the triples (p, q, s) are given by $(p, q, 1)$ (where p, $q \geq 1$), $(p, 2, 2)$ (where $p \geq 2$), $(3, 3, 2)$, $(4, 3, 2)$ and $(5, 3, 2)$ (for type $(p, q, 1)$ the third arm and the relation are redundant) [7].

A finite-dimensional left Λ-module M consists of finite-dimensional vector spaces $M(i)$ for each point i of the quiver, and a linear map $M(\alpha)$ for each arrow $\alpha = \alpha_i$, β_j and γ_k, satisfying the relation

$$M(\gamma_s) \circ \dots \circ M(\gamma_1) = M(\alpha_p) \circ \dots \circ M(\alpha_1) + M(\beta_q) \circ \dots \circ M(\beta_1).$$

The number $\mathrm{rk}(M) = \dim M(\infty) - \dim M(0)$ is called the *rank* of M. Then an indecomposable module is of positive rank (negative rank, rank zero, respectively) if it is *preprojective* (*preinjective*, *regular*, respectively). It is further known that for each indecomposable module M over a domestic canonical algebra $|\,\mathrm{rk}(M)| \leq 6$, moreover the case $|\,\mathrm{rk}(M)| = 6$ appears only for the type $(5, 3, 2)$. The global structure of the module category is also well-known [7]: There is precisely one preprojective component, precisely one preinjective component. All indecomposable preprojective and preinjective modules are exceptional. Recall that an A-module M is called *exceptional* if $\mathrm{End}(M)$ is a division ring and $\mathrm{Ext}^i_A(M, M) = 0$ for all $i \geq 1$. The indecomposable regular modules form tubes, almost all of them homogeneous. If k is algebraically closed, then the family of tubes is parametrized by the projective line over k.

All the modules M with $|\operatorname{rk}(M)| \leq 5$ described in [5] are indecomposable for domestic canonical algebras over an arbitrary field, however the construction given there fails in the case of fields of characteristic 2 for modules of rank 6 and -6 (see the next chapter).

The aim of this paper is to present matrices for the indecomposable preprojective modules of rank 6 over a canonical algebra of type $(5,3,2)$ independently of the characteristic of the field. By dualization one obtains also a description of the indecomposable preinjective modules of rank -6.

A result of Ringel states that for the path algebra $A = kQ$ of each quiver Q any exceptional module can be exhibited using matrices involving as coefficients just 0 and 1 [8]. All indecomposable preprojective and preinjective modules of our construction will satisfy this property, too.

The paper basically follows the idea of [5] and arose when the first author was writing her master thesis at the University of Szczecin.

2. THE MATRICES OF INDECOMPOSABLE RANK 6-MODULES

From now on Λ denotes a canonical algebra of quiver type of type $(5,3,2)$ over an arbitrary field k. In order to describe the indecomposable Λ-modules we are interested in we use the following notations. For natural numbers n and i let I_n be the $n \times n$-identity matrix. Moreover, define

$$X_{n+i,n} = \left[\begin{array}{ccc} & I_n & \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array}\right] \in \mathrm{M}_{n+i,n}(k), \quad Y_{n+i,n} = \left[\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \hline & I_n & \end{array}\right] \in \mathrm{M}_{n+i,n}(k),$$

both having i zero rows of length n. We sometimes denote these matrices by I_*, X_{**} and Y_{**}, respectively, if the format is clear from the context. Furthermore, we define the following matrices

$$Z^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, Z^{(2)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, Z^{(3)} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Z^{(4)} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, Z^{(5)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

In general, if Z is a 6×3-matrix and $n \geq 0$ then we call n-th *enlargement* of Z the matrix

$$\bar{Z}_{n+6,n+3} = \left[\begin{array}{c|ccc} & 1 & & \\ Z & & \ddots & \\ \hline & 1 & & 1 \\ 0 & & \ddots & \\ & & & 1 \end{array}\right] \in \mathrm{M}_{n+6,n+3}(k)$$

with entries 1 on two diagonals, each of length n and 5 zeros between them in each column.

It is well known that each indecomposable preprojective Λ-module is uniquely determined, up to isomorphism, by its dimension vector. Moreover, there are 5 types of indecomposable rank 6-modules (see [5, section 2]). The type depends on the arrow for which the corresponding linear map increases the dimension of the vector spaces by 2, (note that all linear maps for indecomposable preprojective modules are monomorphisms [2, Lemma 4.2] but for rank 6-modules not isomorphisms. Thus the dimension vectors of indecomposable rank 6 modules appear in 5 series.

$$\underline{type 1}\quad \left[\begin{array}{cccccc} & n+2 & n+3 & n+4 & n+5 & \\ n & & n+2 & n+4 & & n+6 \\ & & n+3 & & & \end{array}\right] \qquad \underline{type 2}\quad \left[\begin{array}{cccccc} & n+1 & n+3 & n+4 & n+5 & \\ n & & n+2 & n+4 & & n+6 \\ & & n+3 & & & \end{array}\right]$$

$$\underline{type 3}\quad \left[\begin{array}{cccccc} & n+1 & n+2 & n+4 & n+5 & \\ n & & n+2 & n+4 & & n+6 \\ & & n+3 & & & \end{array}\right] \qquad \underline{type 4}\quad \left[\begin{array}{cccccc} & n+1 & n+2 & n+3 & n+5 & \\ n & & n+2 & n+4 & & n+6 \\ & & n+3 & & & \end{array}\right]$$

$$\underline{type 5}\quad \left[\begin{array}{cccccc} & n+1 & n+2 & n+3 & n+4 & \\ n & & n+2 & n+4 & & n+6 \\ & & n+3 & & & \end{array}\right]$$

Observe that for every type there is an indecomposable module for $n = 0$ which is just a representation of the Dynkin quiver of type E_6 with subspace orientation.

A typical example of a series of indecomposable Λ-modules of rank 6 with dimension vector of type 5 is the following: For $n = 6m + 1,\ m \geq 0$ we define

$$\begin{tikzcd}
 & & k^{n+1} \arrow[r, "X_{n+2,n+1}"] & k^{n+2} \arrow[r, "X_{n+3,n+2}"] & k^{n+3} \arrow[r, "X_{n+4,n+3}"] & k^{n+4} \arrow[dr, "X_{n+6,n+4}"] & \\
M_n : & k^n \arrow[ur, "X_{n+1,n}"] \arrow[rr, "Y_{n+2,n}"] \arrow[drr, "Y_{n+3,n}"'] & & k^{n+2} \arrow[r, "Y_{n+4,n+2}"] & k^{n+4} \arrow[rr, "Y_{n+6,n+4}"] & & k^{n+6} \\
 & & & k^{n+3} \arrow[urrr, "Z_{n+6,n+3}"'] & & &
\end{tikzcd}$$

where $Z_{n+6,n+3}$ is the n-th enlargement of $Z^{(2)}$ (for the proof see section 3).

In [5] in the situation that the characteristic of k is different from 2 indecomposable modules were defined in a similar way, i.e. using matrices X_{**} and Y_{**} and in this case for the rank 6 modules one could take for all types for Z the matrix

$$Z^{(6)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

However, it is easily checked, that working with $Z^{(6)}$ and a field of characteristic 2, for example, the modules M_0 and M_1 for type 5 are decomposable.

Here we solve the problem to give explicit matrices for the rank 6 modules independently of the characteristic. We emphasize that the usual induction procedure, to show that modules of a given series M_n are indecomposable, does not work. For example, one easily calculates that the module M_1 of type 5 defined by the first enlargement of the matrix Z appearing in Gabriel's classification (this is the matrix $Z^{(5)}$ defined above) is a decomposable Λ-module.

We will use the induction $n \to n+6$. Therefore we have to describe for the 5 types matrices for the modules M_n, $0 \leq n \leq 5$. Comparing with the modules given in [5] we only change the shape of the matrix Z, all the other matrices are of the form X_{**} and Y_{**} as in the example considered before. The shape of the matrices Z in the different cases is presented in the table below.

Using this table and the method of enlargement of matrices we obtain 30 series of Λ-modules M_n, $n = 6m + r$, $r = 0, \ldots, 5$, hence a Λ-module for each dimension vector of an indecomposable module of rank 6.

Theorem 1. *The modules of the* 30 *series* M_n, $n = 6m + r$, $r = 0, \ldots, 5$, $m \geq 0$, *constructed by matrices* X_{**} *(respectively* Y_{**} *) of the suitable format for the arrows in the first (respectively second) arm, by* Y_{**} *for the arrow* γ_1 *and by the n-th enlargement of the corresponding matrix* Z *for* M_r *of Table* 1 *for the arrow* γ_2*, are indecomposable.*

	$type1$	$type2$	$type3$	$type4$	$type5$
M_0	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$	$Z^{(4)}$	$Z^{(5)}$
M_1	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(3)}$	$Z^{(2)}$
M_2	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$
M_3	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(4)}$	$Z^{(5)}$
M_4	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$
M_5	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(3)}$	$Z^{(2)}$

TABLE 1. List of matrices Z for indecomposable module of rank 6

3. PROOFS

In order to prove Theorem 1 it is sufficient to show that the constructed modules have endomorphism ring k. This will be done by induction in the same method as in [5]. We illustrate this for the series $M_n,\ n = 6m+1$, of modules of type 5

Let

$$\begin{pmatrix} & B & C & D & E & \\ A & & F & G & & S \\ & & H & & & \end{pmatrix}$$

be an endomorphism of M_n, where $A \in \mathrm{M}_n(k),\ B \in \mathrm{M}_{n+1}(k),\ C \in \mathrm{M}_{n+2}(k),\ D \in \mathrm{M}_{n+3}(k),\ E \in \mathrm{M}_{n+4}(k),\ F \in \mathrm{M}_{n+2}(k),\ G \in \mathrm{M}_{n+4}(k),;\ H \in \mathrm{M}_{n+3}(k),(k),\ S \in \mathrm{M}_{n+6}(k)$.

Dropping the indices we write $X = X_{**}$, $Y = Y_{**}$ and $\bar{Z} = Z_{**}$. Thus we have the following commutativity relations:

$$BX = XA,\ CX = XB,\ DX = XC,\ EX = XD,\ SX = XE,$$
$$FY = YA,\ GY = YF,\ SY = YG,$$
$$HY = YA,\ S\bar{Z} = \bar{Z}H.$$

It follows, that A, B, C, D, E, F and G are nested submatrices of S. Furthermore, the shape of S is shown in Figure 1 where the gray shaded regions indicate zero lines.

This is a consequence of the relations for the arrows of the first and the second arm. Note that the matrix A occurs twice, as left upper corner and as right lower corner of S. Further, from $HY = YA$ we obtain that the matrix A occurs as a right lower corner of H.

In particular, for $m = 0$ it follows that

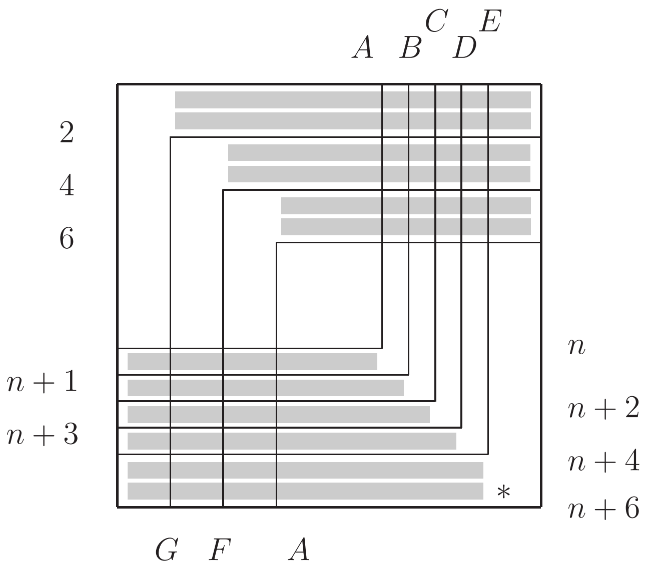

Figure 1. The shape of S

$$S = \begin{bmatrix} s_{11} & s_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & s_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_{33} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & s_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{76} & s_{11} \end{bmatrix}, \; H = \begin{bmatrix} h_{12} & h_{13} & h_{13} & 0 \\ h_{21} & h_{21} & h_{23} & 0 \\ h_{31} & h_{32} & h_{33} & 0 \\ h_{41} & h_{42} & h_{43} & s_{11} \end{bmatrix}.$$

Using this, it is easy to verify that the matrix equation $S\bar{Z} = \bar{Z}H$ implies that H and S (and therefore also B, C, D, E, F, G) are of the form αI_* with the same $\alpha \in k$. Consequently the endomorphism ring of M_1 is isomorphic to k.

Let now $n = 6m + 1$, $m > 0$ and assume that $\operatorname{End}(M_{n-6}) \simeq k$. In the following, if U is a matrix, denote by U' the submatrix of U deleting the last 6 columns and rows.

In order to apply the induction hypothesis we have to show the relations

$$B'X' = X'A',\ C'X' = X'B',\ D'X' = X'C',\ E'X' = X'D',\ S'X' = X'E',$$
$$F'Y' = Y'A',\ G'Y' = Y'F',\ S'Y' = Y'G',$$
$$H'Y' = Y'A',\ S'\bar{Z}' = \bar{Z}'H'.$$

This is true for the relations for the first arm, for example we have

$$SX = \boxed{\begin{array}{c|c} S' & \\ \hline & \end{array}} \cdot \boxed{\begin{array}{c|c} X' & \\ \hline 0 & \end{array}} = \boxed{\begin{array}{c|c} S' \cdot X' & \\ \hline & \end{array}}$$

and by $SX = XE$, and since by Figure 1 all

$$e_{n-1,1} = \cdots = e_{n-1,n-2} = 0 = e_{n,1} = \cdots = e_{n,n-2},$$

this equals

$$XE = \left[\begin{array}{c|cccc} X' & 1 & & & \\ & 0 & 1 & & \\ \hline & & & \ddots & \\ & & & & 1 \\ & & & & 0 \\ & & & & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} E' & \\ \hline 0 \ \cdots \ 0 & \\ 0 \ \cdots \ 0 & \\ \hline & \end{array}\right] = \left[\begin{array}{c|c} X' \cdot E' & \\ \hline & \end{array}\right]$$

All other relations for the first arm follow by similar arguments. The relations for the second arm and $H'Y' = Y'A'$ can be shown in the same way.

We prove now that $S'\bar{Z}' = \bar{Z}'H'$. We have

$$S\bar{Z} = \left[\begin{array}{c|c} S' & \\ \hline & \end{array}\right] \cdot \left[\begin{array}{c|c} \bar{Z}' & \\ \hline 0 & \end{array}\right] = \left[\begin{array}{c|c} S'\bar{Z}' & \\ \hline & \end{array}\right]$$

and for

$$\bar{Z}H = \left[\begin{array}{c|c} \bar{Z}' & * \\ \hline 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} H' & \\ \hline * & \end{array}\right] = \left[\begin{array}{c|c} \bar{Z}'H' & \\ \hline & \end{array}\right]$$

it is sufficient to show that

$$h_{ij} = 0 \text{ for } i = n-2, \ldots, n+3, \text{ and } j = 1, \ldots, n-3.$$

Observe that

$$h_{ij} = (\bar{Z} \cdot H)_{i+3,j} = (S \cdot \bar{Z})_{i+3,j} = \sum_{k=1}^{n+6} s_{i+3,k} \cdot z_{kj}.$$

But $z_{kj} = 0$ for $k = n+1, \ldots, n+6$ and $s_{ik} = 0$ for $i = n+1, \ldots, n+6$ and $k = 1, \ldots, n$, therefore $h_{ij} = 0$. By the induction hypothesis, all the matrices $A', \ldots, S'$ are of the form αI_*, for the same $\alpha \in k$. Then the same follows for the matrices $A, \ldots, S$.

The proofs for the other series of modules are similar. Observe that in the induction step the particular shape of the matrix Z was not important, therefore all the arguments used there can be repeated. Therefore it remains to show that the remaining 29 modules for the beginning of the induction are indecomposable. This can be shown easily as in the case discussed above.

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Received: September 18, 2007