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Evaluation Codes Associated to Complete Bipartite Graphs

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Abstract

In this paper we define the evaluation codes arising from the incidence matrix of a complete bipartite graph. The main parameters of these codes (length, dimension, minimum distance) are computed.

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1 Introduction

It is known that the evaluation codes arising from some subsets of the projective space have been studied in many particular cases (cf. [1], [2], [4], [5], [6], [7], [8], [9], [12], [13], [14], [15], [16]).

In this work we will describe the main parameters of the evaluation codes associated to the toric variety determined by the incidence matrix of a complete bipartite graph .

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2 Evaluation Codes

Let K be a finite field with q elements, let $\mathbb{P}^l_K$ be the $l-$ projective space over K and $X = \{P_1, \ldots, P_s\}$ be a subset of $\mathbb{P}^l_K$. We always use the standard representation for the points in $\mathbb{P}^l_K$, i.e., $P = (0, 0, \ldots, 0, 1, a_i, \ldots, a_l)$. Let $\mathcal{L}$ be a finite dimensional $K-$ linear space of functions which are defined on the set X and take values on K. Thus the evaluation map

$$ev : \mathcal{L} \to K^s,$$
$$ev(f) = (f(P_1), \ldots, f(P_s))$$

defines a $K-$linear code: $C_X = ev(\mathcal{L})$.

Let $S = K[X_0, \ldots, X_l] = \bigoplus_{d \geq 0} S_d$ be the polynomial ring over the finite field K with the natural gradation. If $\mathcal{L} = S_d$ is the $d-$graded homogeneous component of the polynomial ring S, the corresponding linear code $C_X(d) := ev(S_d)$ will be called the evaluation linear code over the set X, which is isomorphic to $S_d/I_X(d)$, where $I_X = \bigoplus_{d \geq 0} I_X(d)$ is the graded vanishing ideal of X. The dimension of these codes is given by the Hilbert function of S/I_X, i.e., $\dim_K C_X(d) = H_X(d)$.

3 Toric Varieties

In this paper we will work with the case where X is a toric variety. Our definition of a toric variety agrees with the given by R.H. Villarreal in [17].

Let $A = (a_{ij})$ be a fixed $m \times (n+1)$ matrix with non negative integer entries a_{ij} and with non-zero columns. Let $K[X_0, \ldots, X_n]$ and $K[t_1, \ldots, t_m]$ the two polynomial rings over K, and φ the graded homomorphism of $K-$algebras

$$\varphi : K[X_0, \ldots, X_n] \to K[t_1, \ldots, t_m]$$

induced by

$$\varphi(X_i) = t_1^{a_{1i}} \cdots t_m^{a_{mi}}$$

The kernel of φ, denoted by I_A, is called the toric ideal associated with the matrix A.

Remark 3.1 *When the field K is algebraically closed, we can use Macaulay 2 (cf. [9]) to compute the toric ideal I_A (cf. [3]).*

The toric variety determined by the matrix A is the subset of the projective space $\mathbb{P}^n_K$ given by

$$X = \{(t_1^{a_{11}} \cdots t_m^{a_{m1}}, \ldots, t_1^{a_{1(n+1)}} \cdots t_m^{a_{m(n+1)}}) \in \mathbb{P}^n_K | \, t_1, \ldots, t_m \in K\}$$

Of course, we take the values of $t_1, \ldots, t_m \in K$ so that they define a point in $\mathbb{P}^n_K$.

4 Complete Bipartite Graphs

Let $K_{m,n}$ be a complete bipartite graph (cf. [10]). The incidence matrix associated to $K_{m,n}$ is the $(m+n)\times(mn)$ matrix $B=(b_{ij})$ with $b_{ij}=1$ if the vertex v_i and the edge a_j are incident and $b_{ij}=0$ otherwise.
In the general case, the toric variety X associated to the incidence matrix of the complete bipartite graph $K_{m,n}$ is given by

$$X=\{(t_1t_{m+1},t_1t_{m+2},\dots,t_1t_{m+n},\,t_2t_{m+1},t_2t_{m+2},\dots,t_2t_{m+n},\dots \\ t_mt_{m+1},t_mt_{m+2},\dots,t_mt_{m+n}:t_i\in K^*\text{ for all } i=1,\dots,m+n\}$$

And in fact, it can be written as

$$X=\{(1,\alpha_1,\alpha_2,\dots,\alpha_{n-1},\,\beta_1,\alpha_1\beta_1,\alpha_2\beta_1,\dots,\alpha_{n-1}\beta_1, \\ \beta_2,\alpha_1\beta_2,\alpha_2\beta_2,\dots,\alpha_{n-1}\beta_2,\dots,\,\beta_{m-1},\alpha_1\beta_{m-1},\alpha_2\beta_{m-1},\dots,\alpha_{n-1}\beta_{m-1}) \\ :\alpha_1,\dots,\alpha_{n-1},\beta_1,\dots,\beta_{m-1}\in K^*\}$$

Let $s=\#(X)$ and consider the following evaluation map

$$\theta:K[Z_{00},\dots,Z_{(m-1)(n-1)}]_d\to K^s$$
$$\theta(f)=(f(P_1,\dots,f(P_s)))$$

where $X=\{P_1,\dots,P_s\}$.

In this case, the evaluation code of order d, $C_X(d)$, associated to the incidence matrix of the complete bipartite graph $K_{m,n}$ is the image of the last evaluation map.

In the next section we will describe the main parameters of these codes.

5 Main Results

In this section we will find the main parameters (length, dimension and minimum distance) of the evaluation codes of order d arising from the incidence matrix of a complete bipartite graph.

The following theorem calculates the length of the given codes.

Theorem 5.1 *With the notation used in the last section, the length of the evaluation code associated to the incidence matrix of the complete bipartite graph $K_{m,n}$ is given by*

$$s=(q-1)^{m+n-2}$$

Proof Let

$$X_1 = \{(1, \alpha_1, \dots, \alpha_{n-1}) : \alpha_1, \dots, \alpha_{n-1} \in K^*\}$$

and

$$X_2 = \{(1, \beta_1, \dots, \beta_{m-1}) : \beta_1, \dots, \beta_{m-1} \in K^*\}$$

Obviously, $\#(X_1) = (q-1)^{n-1}$ and $\#(X_2) = (q-1)^{m-1}$.
Let ψ the Segre map (cf. [11]) restricted to the set $X_1 \times X_2 \subset \mathbb{P}_K^{n-1} \times \mathbb{P}_K^{m-1}$, and with values in $\mathbb{P}_K^{mn-1}$. Therefore, the image of this map is X. Due to the fact that the Segre map in an embedding, we conclude that

$$s = \#(X) = \#(X_1) \cdot \#(X_2) = (q-1)^{m+n-2}$$

and the claim follows. ■

Now, we will find the dimension of these codes. In order to do this, the following notation will be useful: Let H_{X_1} (respectively H_{X_2}) the Hilbert function of the set X_1 (respectively X_2), which gives the dimension of the generalized Reed-Solomon codes determined by the set X_1 (respectively X_2)(cf. [5]). Moreover, let H_X the Hilbert function of the toric variety X which gives the dimension of the evaluation codes that we are working with. The next theorem describes the given dimension.

Theorem 5.2 *The dimension of the evaluation code of order d associated to the incidence matrix of the complete bipartite graph $K_{m,n}$ is given by*

$$dim_K C_X(d) = H_{X_1}(d) \cdot H_{X_2}(d)$$

Proof Let

$$\theta_1 : K[X_0, \dots, X_{n-1}]_d \to K^{s_1}$$
$$\theta_1(g) = (g(Q_1), \dots, g(Q_{s_1}))$$

where $s_1 = (q-1)^{n-1}$ and $X_1 = \{Q_1, \dots, Q_{s_1}\}$. Then $C_{X_1}(d)$ (the generalized Reed-Solomon code of order d associated to X_1) is the image of the last map and

$$C_{X_1}(d) \simeq K[X_0, \dots, X_{n-1}]_d / I_{X_1}(d)$$

In the same way, we define

$$\theta_2 : K[Y_0, \dots, Y_{m-1}]_d \to K^{s_2}$$
$$\theta_2(h) = (h(R_1), \dots, h(R_{s_2}))$$

where $s_2 = (q-1)^{m-1}$ and $X_2 = \{R_1, \dots , R_{s_2}\}$. Then $C_{X_2}(d)$ (the generalized Reed-Solomon code of order d associated to X_2) is the image of the last map and

$$C_{X_2}(d) \simeq K[Y_0, \dots , Y_{m-1}]_d / I_{X_2}(d)$$

The theorem follows from the next isomorphism

$$K[Z_{00}, \dots , Z_{(m-1)(n-1)}]_d / I_X(d) \simeq$$
$$K[X_0, \dots , X_{n-1}]_d / I_{X_1}(d) \bigotimes_K K[Y_0, \dots , Y_{m-1}]_d / I_{X_2}(d)$$

where $\bigotimes_K$ is the tensor product of the corresponding $K-$linear spaces. ∎

Remark 5.3 *In [5] we found the dimension of the generalized Reed-Solomon codes. We obtained that*

$$H_{X_1}(d) = \binom{n-1+d}{d} + \sum_{i=1}^{n-1} (-1)^i \binom{n-1}{i} \binom{n-1+d-i(q-1)}{d-i(q-1)}$$

and

$$H_{X_2}(d) = \binom{m-1+d}{d} + \sum_{i=1}^{m-1} (-1)^i \binom{m-1}{i} \binom{m-1+d-i(q-1)}{d-i(q-1)}$$

and it allows us to compute $\dim_K C_X(d)$.

Corollary 5.4 *The* $a-$*invariant,* a_X*, of the toric variety* X *is given by*

$$a_X = \max \{(n-1)(q-1) - n, (m-1)(q-1) - m\}$$

Proof It is an immediate consequence of the previous theorem and the fact that (cf. [5])

$$a_{X_1} = (n-1)(q-1) - n \qquad a_{X_2} = (m-1)(q-1) - m \quad ∎$$

Finally, we obtain the minimum distance of these codes. Let $\delta_{X_1}(d)$ (respectively $\delta_{X_2}(d)$) the minimum distance of the generalized Reed-Solomon code determined by X_1 (respectively X_2).

Theorem 5.5 *The minimum distance of the evaluation code of order* d *associated to the incidence matrix of the complete bipartite graph* $K_{m,n}$*,* $\delta_X(d)$*, is given by*

$$\delta_X(d) = \delta_{X_1}(d) \cdot \delta_{X_2}(d)$$

Proof Let $f \in K[Z_{00}, \ldots, Z_{(n-1)(m-1)}]_d$ and $X = \{P_{11}, \ldots, P_{s_1 s_2}\}$ where $s_1 = (q-1)^{n-1}$ and $s_2 = (q-1)^{m-1}$. Let $\Upsilon = (f(P_{11}), \ldots, f(P_{s_1 s_2})) \in C_X(d) - \{0\}$. We know that if ψ is the Segre map restricted to $X_1 \times X_2$ (see the proof of the Theorem 5.1) then, for all i, j, we can find $Q_i \in X_1$ and $R_j \in X_2$ so that $\psi(Q_i, R_j) = P_{ij}$. Therefore

$$\Upsilon = (f(X_0Y_0, \ldots, X_{n-1}Y_{m-1})(Q_1, R_1), f(X_0Y_0, \ldots, X_{n-1}Y_{m-1})(Q_{s_1}, R_{s_2}))$$

If we take the polynomial $f(X_0Y_0, \ldots, X_{n-1}Y_{m-1})$, for each $Q_i \in X_1$ (respectively $R_j \in X_2$), f_{Q_i} (respectively f_{R_j}) will mean the previous polynomial evaluated in Q_i (respectively R_j) and then $f_{Q_i} \in K[Y_0, \ldots, Y_{m-1}]_d$ (respectively $f_{R_j} \in K[X_0, \ldots, X_{n-1}]_d$).

Let $\Upsilon_i = (f_{Q_i}(R_1), \ldots, f_{Q_i}(R_{s_2})) \in C_{X_2}(d)$ for all $i = 1, \ldots, s_1$ and let $s_3 = \#\{i : \Upsilon_i \neq 0\}$. Since the Hamming weight of Υ_i ($w(\Upsilon_i)$) is such that $w(\Upsilon_i) \geq \delta_{X_2}(d)$ for all i with $\Upsilon_i \neq 0$, we conclude that $w(\Upsilon) \geq s_3 \delta_{X_2}(d)$.

On the other hand, let $\Lambda_j = (f_{R_j}(Q_1), \ldots, f_{R_j}(Q_{s_1})) \in C_{X_1}(d)$ for all $j = 1, \ldots, s_2$. If j is such that $\Lambda_j \neq 0$, then whenever $s_3 < \delta_{X_1}(d)$ we have that $w(\Lambda_j) \leq s_3 < \delta_{X_1}(d)$. Therefore $w(\Upsilon) \geq \delta_{X_1}(d) \cdot \delta_{X_2}(d)$.

The claim follows immediately because we can find a word in $C_X(d)$ with Hamming weight equal to $\delta_{X_1}(d) \cdot \delta_{X_2}(d)$. ■

6 An Example

In order to illustrate the main results of this paper we will give an specific example. The computations were made with Macaulay 2 (cf. [9]).

Let the complete bipartite graph $K_{2,3}$ shown in the figure 1.

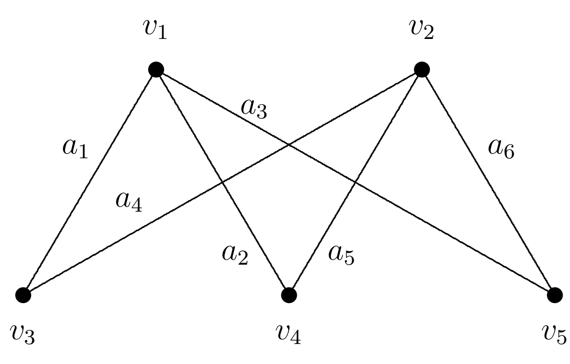

Figure 1: $K_{2,3}$

Its incidence matrix is a 5×6 matrix given by

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

We will work with a finite field K with 5 elements.

The toric variety X has $(5-1)^3 = 64$ points and this is the length of the code $C_X(d)$. We do not include its vanishing ideal because it has too many generators. The Hilbert function, which gives the dimension of the corresponding codes is described in the next table

$H_X(0)$	$H_X(1)$	$H_X(2)$	$H_X(3)$	$H_X(4)$	$H_X(5)$	$H_X(6)$
1	6	18	40	52	60	64

In fact, the Hilbert series is given by

$$F_X(t) = \frac{1-3t^2+2t^3-23t^4+72t^5-81t^6+38t^7-2t^8-12t^9+12t^{10}-4t^{11}}{(1-t)^6}$$

and, of course, the $a-$invariant is 5.

Moreover, if we consider the code $C_X(3)$, we can compute its minimum distance. In this case $\delta_{X_1}(3) = 4$ and $\delta_{X_2}(3) = 1$ (cf. [5]). Therefore, $\delta_X(3) = 4$.

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