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The Green-Kehayopulu Relations in le-Γ-Semigroups[1]

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Abstract

We can see that any semigroup can be considered as a Γ-semigroup. The concept of the Green-Kehayopulu relations in le-semigroups was introduced in 2002 by Petro and Pasku [3]. In this paper, we introduce the concept of the Green-Kehayopulu relations in le-Γ-semigroups mimics the definition of the Green-Kehayopulu relations in le-semigroups. We show that, an $\mathcal{H}_\gamma$-class of an le-Γ-semigroup M satisfies Green's condition if and only if it contains a γ-idempotent and an $\mathcal{H}_\gamma$-class of an le-Γ-semigroup M is a subgroup of $< M_\gamma, \circ >$ if and only if it consists of a single idempotent.

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1 Introduction and Preliminaries

In 2002, Petraq Petro and Elton Pasku [3] introduced the concept of the Green-Kehayopulu relations in le-semigroups and showed that a nonsingleton $\mathcal{H}$-class

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cannot be a subgroup and an $\mathcal{H}$-class satisfying "Green's condition" need not constitute a subsemigroup.

In this paper, we introduce the concept of the Green-Kehayopulu relations in le-Γ-semigroups and give necessary and sufficient conditions in order that an $\mathcal{H}_\gamma$-class of le-Γ-semigroup M is a subgroup or a subsemigroup of $< M_\gamma, \circ >$.

To present the main results we first recall the definition of a Γ-semigroup which is important here.

Let M and Γ be any two nonempty sets. M is called a Γ-*semigroup* [5] if there exists a mapping $M \times \Gamma \times M \longrightarrow M$, written as $(a, \gamma, b) \longrightarrow a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. For any $\gamma \in \Gamma$, an element x of a Γ-semigroup M is said to be a γ-*idempotent* if $x\gamma x = x$ [6]. For a Γ-semigroup M and any $\gamma \in \Gamma$, if we define $a \circ b = a\gamma b$ for all $a, b \in M$, then M becomes a semigroup. We denote this semigroup by M_γ [6].

Examples of Γ-semigroups can be seen in [1, 4] and [5], respectively.

The following definitions in this paper are introduced analogously to some definitions in [3].

A Γ-semigroup M is called an le-Γ-*semigroup* if $< M; \vee, \wedge >$ is a lattice with a greatest element (the element is always denoted by e below) [2] and for any $a, b, c \in M$ and $\gamma \in \Gamma$,

$$c\gamma(a \vee b) = c\gamma a \vee c\gamma b \text{ and } (a \vee b)\gamma c = a\gamma c \vee b\gamma c.$$

Throughout this paper M will stand for an le-Γ-semigroup. We shall consider the usual order relation $\leq$ on M defined by for any $a, b \in M$, $a \leq b$ if and only if $a \vee b = b$. Then we can show that for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$. Hence we have known that ordered Γ-semigroups (some author called po-Γ-semigroup) are a generalization of le-Γ-semigroups. For any $\gamma \in \Gamma$, let the mappings $l_\gamma, r_\gamma : M \longrightarrow M$ be defined by for any $x \in M$,

$$l_\gamma(x) = e\gamma x \vee x \text{ and } r_\gamma(x) = x\gamma e \vee x.$$

Then we define equivalence relations on M as follows:

$$\begin{aligned} \mathcal{L}_\gamma &:= \{(x, y) \in M \times M : l_\gamma(x) = l_\gamma(y)\}, \\ \mathcal{R}_\gamma &:= \{(x, y) \in M \times M : r_\gamma(x) = r_\gamma(y)\}, \\ \mathcal{H}_\gamma &:= \mathcal{L}_\gamma \cap \mathcal{R}_\gamma. \end{aligned}$$

We shall call the equivalences $\mathcal{L}_\gamma, \mathcal{R}_\gamma$ and $\mathcal{H}_\gamma$ the *Green-Kehayopulu relations* of M. An element x of M is said to be a γ-*left ideal* (γ-*right ideal*) *element* if

$l_\gamma(x) = x$ ($r_\gamma(x) = x$) and a γ-*ideal element* if it is both a γ-left ideal element and a γ-right ideal element; it is called a γ-*quasi-ideal element* if $e\gamma x \wedge x\gamma e \leq x$. An element x of M is said to be a γ-*regular element* if $x \leq x\gamma e\gamma x$ and a γ-*intra-regular element* if $x \leq e\gamma x\gamma x\gamma e$. An $\mathcal{H}_\gamma$-class H of M satisfying Green's condition if there exist elements a and b of M such that $a\gamma b \in H$.

Before the characterizations of the $\mathcal{H}_\gamma$-class of M for the main results, we give auxiliary results which are necessary in what follows.

Lemma 1.1 *For each $x \in M$ and $\gamma \in \Gamma$,*

$$l_\gamma(l_\gamma(x)) = l_\gamma(x) \text{ and } r_\gamma(r_\gamma(x)) = r_\gamma(x).$$

Proof. From the definition of the mapping l_γ it follows that $l_\gamma(l_\gamma(x)) = l_\gamma(e\gamma x \vee x) = e\gamma(e\gamma x \vee x) \vee e\gamma x \vee x = e\gamma e\gamma x \vee e\gamma x \vee e\gamma x \vee x = e\gamma e\gamma x \vee e\gamma x \vee x$. Since e is the greatest element in M, we also have $e\gamma e \leq e$. Thus $e\gamma e\gamma x \leq e\gamma x$, so $e\gamma e\gamma x \vee e\gamma x = e\gamma x$. Hence $l_\gamma(l_\gamma(x)) = e\gamma x \vee x = l_\gamma(x)$. By symmetry, $r_\gamma(r_\gamma(x)) = r_\gamma(x)$. □

Lemma 1.2 *If an element a of M is a γ-left ideal element and an element b of M is a γ-right ideal element, then $a \wedge b$ is a γ-quasi-ideal element.*

Proof. Assume that a is a γ-left ideal element and b is a γ-right ideal element of M. Then $e\gamma a \vee a = l_\gamma(a) = a$ and $b\gamma e \vee b = r_\gamma(b) = b$, so $e\gamma a \leq a$ and $b\gamma e \leq b$. Hence $e\gamma(a \wedge b) \wedge (a \wedge b)\gamma e \leq e\gamma a \wedge b\gamma e \leq a \wedge b$. Therefore $a \wedge b$ is a γ-quasi-ideal element. □

Lemma 1.3 *For each $x \in M$ and $\gamma, \beta \in \Gamma$,*

$$l_\beta(l_\beta(x) \wedge r_\gamma(x)) = l_\beta(x) \text{ and } r_\gamma(l_\beta(x) \wedge r_\gamma(x)) = r_\gamma(x).$$

Proof. Since $x = x \wedge x \leq l_\beta(x) \wedge r_\gamma(x) \leq l_\beta(x)$, it follows from Lemma 1.1 that $l_\beta(x) \leq l_\beta(l_\beta(x) \wedge r_\gamma(x)) \leq l_\beta(l_\beta(x)) = l_\beta(x)$. Hence $l_\beta(l_\beta(x) \wedge r_\gamma(x)) = l_\beta(x)$. By symmetry, $r_\gamma(l_\beta(x) \wedge r_\gamma(x)) = r_\gamma(x)$. □

Lemma 1.4 *Each $\mathcal{H}_\gamma$-class H of M has a greatest element which is equal to $l_\gamma(a) \wedge r_\gamma(a)$ where a is an arbitrary element in H.*

Proof. Let a be an element of the $\mathcal{H}_\gamma$-class H of M. By Lemma 1.3, we have $(l_\gamma(a) \wedge r_\gamma(a), a) \in \mathcal{L}_\gamma$ and $(l_\gamma(a) \wedge r_\gamma(a), a) \in \mathcal{R}_\gamma$. Thus $(l_\gamma(a) \wedge r_\gamma(a), a) \in \mathcal{H}_\gamma$, so $l_\gamma(a) \wedge r_\gamma(a) \in H$. Now let any $x \in H$. Then $(x, a) \in \mathcal{H}_\gamma = \mathcal{L}_\gamma \cap \mathcal{R}_\gamma$, this implies that $x \leq l_\gamma(x) = l_\gamma(a)$ and $x \leq r_\gamma(x) = r_\gamma(a)$. Hence $x \leq l_\gamma(a) \wedge r_\gamma(a)$, so $l_\gamma(a) \wedge r_\gamma(a)$ is a greatest element of H. □

Lemmas 1.1 and 1.2 imply that for each element a of M, the meet $l_\gamma(a) \wedge r_\gamma(a)$ is a γ-quasi-ideal element. Lemma 1.4 implies that for each element a of the $\mathcal{H}_\gamma$-class H, $l_\gamma(a) \wedge r_\gamma(a)$ is a greatest element of H. We call the element $l_\gamma(a) \wedge r_\gamma(a)$ the *representative γ-quasi-ideal element* of the $\mathcal{H}_\gamma$-class of a; the representative γ-quasi-ideal element of an $\mathcal{H}_\gamma$-class H will be denoted by q_H. From Lemma 1.4, the following properties of q_H hold.

(1) $q_H \in H$.

(2) For each $x \in H$, $l_\gamma(x) \wedge r_\gamma(x) = q_H$; in particular, $l_\gamma(q_H) \wedge r_\gamma(q_H) = q_H$.

(3) For each $x \in H$, $x \leq q_H$.

Lemma 1.5 *If elements x and y of M are $\mathcal{R}_\gamma$-related (resp. $\mathcal{L}_\gamma$-related), then $x\gamma e = y\gamma e$ (resp. $e\gamma x = e\gamma y$).*

Proof. Assume that $(x, y) \in \mathcal{R}_\gamma$. Then $r_\gamma(x) = r_\gamma(y)$, so $x\gamma e \vee x = y\gamma e \vee y$. This implies that $x\gamma e\gamma e \vee x\gamma e = (x\gamma e \vee x)\gamma e = (y\gamma e \vee y)\gamma e = y\gamma e\gamma e \vee y\gamma e$. Since $e\gamma e \leq e$, $x\gamma e\gamma e \leq x\gamma e$ and $y\gamma e\gamma e \leq y\gamma e$. Hence $x\gamma e = x\gamma e\gamma e \vee x\gamma e = y\gamma e\gamma e \vee y\gamma e = y\gamma e$. Similarly, $(x, y) \in \mathcal{L}_\gamma$ implies $e\gamma x = e\gamma y$. □

Lemma 1.6 *If H is an $\mathcal{H}_\gamma$-class of M and $x \in H$, then $e\gamma x \wedge x\gamma e = e\gamma q_H \wedge q_H\gamma e$.*

Proof. Assume that H is an $\mathcal{H}_\gamma$-class of M and $x \in H$. Then $(x, q_H) \in \mathcal{H}_\gamma$. It follows from Lemma 1.5 that $e\gamma x = e\gamma q_H$ and $x\gamma e = q_H\gamma e$. Hence $e\gamma x \wedge x\gamma e = e\gamma q_H \wedge q_H\gamma e$. □

2 Main Results

In this section, we characterize the relationship between the $\mathcal{H}_\gamma$-classes of M satisfying Green's condition and the semigroup $< M_\gamma, \circ >$ and give some conditions which ensure that an $\mathcal{H}_\gamma$-class of M forms a subgroup or a subsemigroup of the semigroup $< M_\gamma, \circ >$.

The following theorems collect several properties that hold in every $\mathcal{H}_\gamma$-class of M satisfying Green's condition.

Theorem 2.1 *Let H be an $\mathcal{H}_\gamma$-class of M satisfying Green's condition and $q = q_H$. Then we have the following statements:*

(*a*) *$q\gamma q \in H$ and $q = e\gamma q \wedge q\gamma e$.*

(*b*) *The element q is the only γ-quasi-ideal element in H.*

(*c*) *If $x, y \in H$, then $y \leq e\gamma x$ and $y \leq x\gamma e$.*

(*d*) *For each integer $n \geq 2$, let $\gamma_1, \gamma_2, ..., \gamma_{n-1} \in \{\gamma\}$. Then $q\gamma q = q\gamma e\gamma q = q\gamma_1 q\gamma_2 q...q\gamma_{n-1}q$; in particular, $q\gamma q$ is a γ-idempotent.*

(*e*) *Every element of H is a γ-intra-regular element.*

(*f*) *The element q is a γ-idempotent if and only if q is a γ-regular element in which case every element of H is a γ-regular element.*

Proof. (*a*) Since H satisfies Green's condition, there exist $b, c \in H$ such that $b\gamma c \in H$. Since $b, c \in H$, we have $b \leq q$ and $c \leq q$. Thus $b\gamma c \leq q\gamma q \leq q\gamma e$, this implies that $r_\gamma(b\gamma c) \leq r_\gamma(q\gamma q) \leq r_\gamma(q\gamma e)$. Since $(b\gamma c, q) \in \mathcal{H}_\gamma$, $(b\gamma c, q) \in \mathcal{R}_\gamma$. Thus $r_\gamma(b\gamma c) = r_\gamma(q)$. On the other hand, since $e\gamma e \leq e$, we have $r_\gamma(q\gamma e) = q\gamma e\gamma e \vee q\gamma e = q\gamma e \leq q\gamma e \vee q = r_\gamma(q)$. Hence $r_\gamma(q) = r_\gamma(b\gamma c) \leq r_\gamma(q\gamma q) \leq r_\gamma(q\gamma e) = q\gamma e \leq r_\gamma(q)$, so $r_\gamma(q) = r_\gamma(q\gamma q) = q\gamma e$. By symmetry, $l_\gamma(q) = l_\gamma(q\gamma q) = e\gamma q$. Therefore $(q, q\gamma q) \in \mathcal{H}_\gamma$, so $q\gamma q \in H$. It follows that $q = l_\gamma(q) \wedge r_\gamma(q) = e\gamma q \wedge q\gamma e$.

(*b*) By (*a*), q is a γ-quasi-ideal element in H. Now let t be any γ-quasi-ideal element in H. By (*a*) and Lemma 1.6, we have $t \leq q = e\gamma q \wedge q\gamma e = e\gamma t \wedge t\gamma e \leq t$. Hence $t = q$, so we conclude that q is the only γ-quasi-ideal element in H.

(*c*) Let any $x, y \in H$. By (*a*) and Lemma 1.6, we have $y \leq q = e\gamma q \wedge q\gamma e = e\gamma x \wedge x\gamma e$. Hence $y \leq e\gamma x$ and $y \leq x\gamma e$.

(*d*) By (*a*), $q = e\gamma q \wedge q\gamma e \leq q\gamma e$. Thus $q\gamma q \leq q\gamma e\gamma q$. Since $e\gamma q \leq e$, $q\gamma e\gamma q \leq q\gamma e$. Similarly, since $q\gamma e \leq e$, $q\gamma e\gamma q \leq e\gamma q$. Thus $q\gamma e\gamma q \leq e\gamma q \wedge q\gamma e = q$. Hence $q\gamma q\gamma e\gamma q \leq q\gamma q$. By (*a*), we get $(q\gamma q, q) \in \mathcal{R}_\gamma$. By Lemma 1.5, $q\gamma e = q\gamma q\gamma e$ and it follows that $q\gamma e\gamma q = q\gamma q\gamma e\gamma q$. Hence $q\gamma q \leq q\gamma e\gamma q$ and $q\gamma e\gamma q \leq q\gamma q$, so $q\gamma q = q\gamma e\gamma q$. Now let any integer $k \geq 2$ and $\gamma_1, \gamma_2, ..., \gamma_{k-1} \in \{\gamma\}$ be such that $q\gamma_1 q\gamma_2 q...q\gamma_{k-1}q = q\gamma q$. Then $q\gamma_1 q\gamma_2 q...q\gamma_{k-1}q\gamma q = q\gamma q\gamma q = q\gamma(q\gamma e\gamma q) = (q\gamma q\gamma e)\gamma q = q\gamma e\gamma q = q\gamma q$. In particular, $(q\gamma q)\gamma(q\gamma q) = q\gamma q$. Hence $q\gamma q$ is a γ-idempotent.

(*e*) Let any $x \in H$. Then $x \leq q$. By (*a*), we get $q \leq e\gamma q$ and $q \leq q\gamma e$. Thus $x \leq e\gamma q \leq e\gamma q\gamma e$. By (*a*), we get $(q\gamma q, q) \in \mathcal{R}_\gamma$. By Lemma 1.5, $q\gamma e = q\gamma q\gamma e$. This implies that $x \leq e\gamma q\gamma e = e\gamma(q\gamma q\gamma e) = (e\gamma q)\gamma(q\gamma e)$. Since $(x, q) \in \mathcal{H}_\gamma$, it follows from Lemma 1.5 that $e\gamma q = e\gamma x$ and $q\gamma e = x\gamma e$. Hence $x \leq e\gamma x\gamma x\gamma e$, so we conclude that x is a γ-intra-regular element.

(*f*) Assume that $q = q\gamma q$. By (*d*), $q\gamma q = q\gamma e\gamma q$. Thus $q = q\gamma e\gamma q$, so q is a γ-regular element. If $x \in H$, then $x \leq q$. Since $(x, q) \in \mathcal{H}_\gamma$, it follows from Lemma 1.5 that $e\gamma q = e\gamma x$ and $q\gamma e = x\gamma e$. Hence $x \leq q = (q\gamma e)\gamma q = (x\gamma e)\gamma q = x\gamma(e\gamma q) = x\gamma e\gamma x$. Therefore x is a γ-regular element.

Conversely, assume that $q \leq q\gamma e\gamma q$. By (*d*), $q\gamma q = q\gamma e\gamma q$. Thus $q \leq q\gamma q$. By (*a*), $q\gamma q \in H$. Thus $q\gamma q \leq q$. Hence $q = q\gamma q$, so we conclude that q is a γ-idempotent.

Therefore we complete the proof of the theorem. □

Using the Theorems 2.1 (a) and 2.1 (d), we have Corollary 2.2.

Corollary 2.2 *An $\mathcal{H}_\gamma$-class H of M satisfies Green's condition if and only if it contains a γ-idempotent.*

Theorem 2.3 *An $\mathcal{H}_\gamma$-class H of M is a subgroup of $< M_\gamma, \circ >$ if and only if it consists of a single idempotent.*

Proof. Assume that H is a subgroup of M_γ and let $q = q_H$. Then $q\gamma q = q \circ q \in H$, so $q\gamma q \leq q$. Denote by i the identity element of H. Then $i \leq q$, so $q \circ q = q\gamma q \leq q = q \circ i = q\gamma i \leq q\gamma q = q \circ q$. Hence $q \circ q = q$, so we conclude that $q = i$. Now let t be an arbitrary element of H. We denote by t^{-1} the inverse element of t in H. Then $t^{-1} \leq q$, so $q = i = t \circ t^{-1} = t\gamma t^{-1} \leq t\gamma q = t \circ q = t \circ i = t$. On the other hand, $t \leq q$. Therefore $t = q$, so we conclude that H consists of a single idempotent.

The converse is obvious. □

Theorem 2.4 *Let H be an $\mathcal{H}_\gamma$-class of M and $q = q_H$. Then the following statements are equivalent:*

(a) *An $\mathcal{H}_\gamma$-class H is a subsemigroup of $< M_\gamma, \circ >$.*

(b) *If $x \in H$, then $x\gamma x \in H$.*

(c) *An $\mathcal{H}_\gamma$-class H satisfies Green's condition and $x\gamma q = q\gamma q = q\gamma x$ for every $x \in H$.*

Proof. Since H is a subsemigroup of M_γ, we immediately have $x\gamma x = x \circ x \in H$ for all $x \in H$. Therefore (a) implies (b). Let any $x \in H$. Then $x\gamma x \in H$, so H satisfies Green's condition and $(x, x\gamma x) \in \mathcal{H}_\gamma$. By Lemma 1.5, $e\gamma x = e\gamma x\gamma x$ and $x\gamma e = x\gamma x\gamma e$. Similarly, since $(x, q) \in \mathcal{H}_\gamma$, we get $e\gamma x = e\gamma q$ and $x\gamma e = q\gamma e$. By Theorem 2.1 (d), $q\gamma q = q\gamma e\gamma q$. Hence $x\gamma q\gamma q = x\gamma(q\gamma e\gamma q) = x\gamma(q\gamma e)\gamma q = x\gamma(x\gamma e)\gamma q = (x\gamma x\gamma e)\gamma q = (x\gamma e)\gamma q = (q\gamma e)\gamma q = q\gamma q$. Similarly, $q\gamma q\gamma x = q\gamma q$. Since $x, q\gamma q \in H$, we have $x \leq q$ and $q\gamma q \leq q$. Hence $q\gamma q = x\gamma q\gamma q \leq x\gamma q \leq q\gamma q$, so we conclude that $x\gamma q = q\gamma q$. Similarly, $q\gamma x = q\gamma q$. Thus (b) implies (c). Let any $x, y \in H$. Then $(y, q) \in \mathcal{H}_\gamma$, so $(y, q) \in \mathcal{R}_\gamma$. Thus $r_\gamma(y) = r_\gamma(q)$, so $y\gamma e \vee y = q\gamma e \vee q$. Hence $r_\gamma(x\gamma y) = x\gamma y\gamma e \vee x\gamma y = x\gamma(y\gamma e \vee y) = x\gamma(q\gamma e \vee q) = x\gamma q\gamma e \vee x\gamma q = r_\gamma(x\gamma q)$. Since $x \in H$, $x\gamma q = q\gamma q$. This implies that $r_\gamma(x\gamma y) = r_\gamma(q\gamma q)$. By Theorem 2.1 (a), $q\gamma q \in H$. It follows that $r_\gamma(q\gamma q) = r_\gamma(q)$. Hence $r_\gamma(x\gamma y) = r_\gamma(q)$, so $(x\gamma y, q) \in \mathcal{R}_\gamma$. Similarly, since $(y, q) \in \mathcal{L}_\gamma$, we have $(x\gamma y, q) \in \mathcal{L}_\gamma$. We conclude that $(x\gamma y, q) \in \mathcal{H}_\gamma$, so

$x \circ y = x\gamma y \in H$. Therefore H is a subsemigroup of M_γ, so we have that (c) implies (a).

Hence the theorem is now completed. $\square$

As a consequence of Theorem 2.4, we immediately have Corollary 2.5.

Corollary 2.5 *If H is an $\mathcal{H}_\gamma$-class of M and $q_H\gamma x = q_H = x\gamma q_H$ for all $x \in H$, then H is a subsemigroup of $< M_\gamma, \circ >$.*

Lemma 2.6 *If H is an $\mathcal{H}_\gamma$-class of M satisfying Green's condition and $q = q_H$ is a γ-ideal element, then $q\gamma x = q = x\gamma q$ for all $x \in H$.*

Proof. Assume that H is an $\mathcal{H}_\gamma$-class of M satisfying Green's condition and $q = q_H$ is a γ-ideal element. Then $l_\gamma(q) = q$ and $r_\gamma(q) = q$, so $e\gamma q \leq q$ and $q\gamma e \leq q$. By Theorem 2.1 (c), we have $q \leq e\gamma q$ and $q \leq q\gamma e$. This implies that $e\gamma q = q = q\gamma e$. By Theorem 2.1 (a), $q\gamma q \in H$. Thus $(q, q\gamma q) \in \mathcal{L}_\gamma$, it follows from Lemma 1.5 that $e\gamma q = e\gamma q\gamma q$. Therefore $q\gamma e\gamma q = (e\gamma q)\gamma q = e\gamma q = q$. Now let x be an arbitrary element of H. By Lemma 1.5, we have $e\gamma x = e\gamma q$ and $x\gamma e = q\gamma e$. Hence $x\gamma q = x\gamma(e\gamma q) = (x\gamma e)\gamma q = (q\gamma e)\gamma q = q$ and $q\gamma x = (q\gamma e)\gamma x = q\gamma(e\gamma x) = q\gamma(e\gamma q) = q$. Therefore $q\gamma x = q = x\gamma q$ for all $x \in H$.

Hence the proof of the lemma is completed. $\square$

Immediately from Corollary 2.5 and Lemma 2.6, we have Corollary 2.7.

Corollary 2.7 *If H is an $\mathcal{H}_\gamma$-class of M satisfying Green's condition and q_H is a γ-ideal element, then H is a subsemigroup of $< M_\gamma, \circ >$.*

Corollary 2.8 *An $\mathcal{H}_\gamma$-class H of the greatest element e of M is a subsemigroup of $< M_\gamma, \circ >$ if and only if e is a γ-idempotent.*

Proof. Assume that an $\mathcal{H}_\gamma$-class H of the greatest element e of M is a subsemigroup of M_γ. Then $e\gamma e = e \circ e \in H$, so H satisfies Green's condition. Since $e \in H$, $e \leq q_H$. Thus $q_H = e$. Since $e \leq e\gamma e \vee e = l_\gamma(e) = r_\gamma(e) \leq e$, we have $l_\gamma(e) = e = r_\gamma(e)$. Hence e is a γ-ideal element. By Lemma 2.6, $e\gamma x = e = x\gamma e$ for all $x \in H$. Hence $e = e\gamma e$, so e is a γ-idempotent.

Conversely, assume that e is a γ-idempotent in an $\mathcal{H}_\gamma$-class H. Then $e\gamma e = e \in H$, so H satisfies Green's condition. By the above proof, $q_H = e$ and e is a γ-ideal element. It follows from Corollary 2.7 that H is a subsemigroup of M_γ.

Hence the proof is completed. $\square$

Theorem 2.9 *Let H be an $\mathcal{H}_\gamma$-class of M such that its representative γ-quasi-ideal element $q = q_H$ is minimal in the set of all γ-quasi-ideal elements of M. Then $H = \{x \in M : x \leq q\}$ is a subsemigroup of $< M_\gamma, \circ >$.*

Proof. If $x \in H$, then $x \leq q$. Now assume that x is an element of M such that $x \leq q$. Then $l_\gamma(x) \wedge r_\gamma(x) \leq l_\gamma(q) \wedge r_\gamma(q) = q$. By Lemmas 1.1 and 1.2, $l_\gamma(x) \wedge r_\gamma(x)$ is a γ-quasi-ideal element. Since q is a minimal γ-quasi-ideal element, $l_\gamma(x) \wedge r_\gamma(x) = q$. Thus $q \leq l_\gamma(x)$ and $q \leq r_\gamma(x)$. By Lemma 1.1, we have $l_\gamma(q) \leq l_\gamma(l_\gamma(x)) = l_\gamma(x)$ and $r_\gamma(q) \leq r_\gamma(r_\gamma(x)) = r_\gamma(x)$. Since $x \leq q$, we have $l_\gamma(x) \leq l_\gamma(q)$ and $r_\gamma(x) \leq r_\gamma(q)$. Hence $l_\gamma(x) = l_\gamma(q)$ and $r_\gamma(x) = r_\gamma(q)$, so $(x, q) \in \mathcal{L}_\gamma \cap \mathcal{R}_\gamma = \mathcal{H}_\gamma$. Therefore $x \in H$, so we conclude that $H = \{x \in M : x \leq q\}$. Now let x be an arbitrary element of H. Then $x \leq q$. Since $x \leq e$, we have $x\gamma x \leq e\gamma q \wedge q\gamma e \leq l_\gamma(q) \wedge r_\gamma(q) = q$. This implies that $x\gamma x \in H$. It follows from Theorem 2.4 that H is a subsemigroup of M_γ.

Therefore the proof of the theorem is completed. □

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