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A New Characterization of the p-Adic Spectrum

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Abstract

The main purpose of this paper is to give a new characterization of the p-adic spectrum of the polynomial ring $\mathbb{Q}_p[X_1, \dots, X_m]$ in m indeterminates over the p-adic number field $\mathbb{Q}_p$ comparable to the real case. Using this new characterization, we prove a relationship between the dimension of a p-adic semi-algebraic set and that of the constructible set of the p-adic spectrum of $\mathbb{Q}_p[X_1, \dots, X_m]$ associated to this set by the p-adic tilde identification.

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0 Introduction

The *real spectrum* of a ring is a powerful and useful tool in both real algebraic and analytic geometry. It can be used to understand and prove many important results. It was introduced by Cost-Roy in [6]. For a more detailed study of this notion, we refer the interested reader to [4] and [1]. Let us recall here that a point of the real spectrum of the polynomial ring $\mathbb{R}[X_1, \dots, X_m]$ over $\mathbb{R}$ can be regarded as an ordered pair $(I, \leq)$, where I is a real prime ideal of $\mathbb{R}[X_1, \dots, X_m]$ and $\leq$ a linear order on the residue field $k(I)$ of I. But to give a linear order $\leq$ on $k(I)$ is equivalent to give its positive cone

$$\{x \in k(I) |\, 0 \leq x\} = \{x \in k(I) |\, x = y^2 \text{ and } y \in R\} = k(I) \cap R^{(2)},$$

where R is a real closed field. Let us remark that

$$k(I) \cap R^{(2n)} = k(I) \cap R^{(2)} \quad \text{and} \quad k(I) \cap R^{(2n+1)} = k(I) \quad \text{for all } n \geq 1.$$

Thus the subset $k(I) \cap R^{(2)}$ is sufficient to determine a linear order on $k(I)$. Therefore a point of the real spectrum of $\mathbb{R}[X_1, \dots, X_m]$ may be viewed as an ordered pair $(I,\, k(I) \cap R^{(2)})$, where I is a real prime ideal of $\mathbb{R}[X_1, \ldots, X_m]$ and R is a real closed extension field of the residue field $k(I)$ of I. For further details on the theory of real closed fields, we refer the reader to [4] and [9]

Let p be a fixed prime number. The strong and well-known analogy between the theory of real closed fields and that of p-adically closed fields has motivated many mathematicians to translate results and proofs from the real case to the p-adic one. For instance, Macintyre's theorem was shown in [8] to give a p-adic analogue of Tarski's theorem. In the same way, we are interested here by the *p-adic spectrum* of the polynomial ring $\mathbb{Q}_p[X_1, \ldots, X_m]$ over $\mathbb{Q}_p$. More precisely, we will give a new characterization of the points of the *p-adic spectrum* of the polynomial ring $\mathbb{Q}_p[X_1, \ldots, X_m]$ over $\mathbb{Q}_p$ comparable to that of the points of real spectrum of the polynomial ring $\mathbb{R}[X_1, \ldots, X_m]$ as remarked above.

The paper is organized as follows. In section 1 we give briefly some basic facts about p-adic algebra used later. Section 2 is devoted to the p-adic spectrum; first we recall briefly the construction of the p-adic spectrum and then we give some elementary results which are mostly well-known. In section 3 we give the main result of the paper (see Theorem 3.2). Finally, in section 4 as an illustration of the new characterization, we establishe a relationship between the dimension of a p-adic semi-algebraic set and that of the constructible set of the p-adic spectrum of $\mathbb{Q}_p[X_1, \ldots, X_m]$ associated to this set by the p-adic tilde identification.

1 Preliminary notes

Let us first introduce some notations and terminologies. Throughout this paper, any ring is assumed to be a commutative ring with unit. We will denote by $\underline{x}$ the m-tuple $(x_1, \ldots, x_m)$, with m from $\mathbb{N}^*$.

Let K be a p-adically closed field, i.e. a p-adically closed field of p-rank 1 in the sense of [10]. For example, the p-adic number field $\mathbb{Q}_p$ is a p-adically closed field. We will denote by $K^{*(n)}$ (resp. $K^{(n)}$) the subset of K defined as follows

$$K^{*(n)} = \{x \in K^* \mid \exists\, y \in K^* \;\; x = y^n\}$$

$$\big(\text{resp. } K^{(n)} = \big\{x \in K \mid x = 0 \;\text{ or }\; x \in K^{*(n)}\big\}\big)\cdot$$

We denote by $K[\underline{X}] = K[X_1, \ldots, X_m]$ the polynomial ring over K, and by $K(\underline{X})$ the quotient field of $K[\underline{X}]$. The p-adic Kochen operator is defined by:

$$\gamma(X) = \frac{1}{p} \cdot \frac{X^p - X}{(X^p - X)^2 - 1}.$$

The p-adic Kochen ring is the subring of $K(\underline{X})$ defined by:

$$\Lambda = \left\{ \frac{t}{1 + p\, s} \;\middle|\; t, s \in \mathbb{Z}[\gamma(K(\underline{X}))] \right\}\cdot$$

We denote by $\Lambda \cdot K[\underline{X}]$ the subring of $K(\underline{X})$ defined by:

$$\Lambda \cdot K[\underline{X}] = \left\{ \frac{t}{1+ps} \middle| \; t \in K[\underline{X}, \gamma(K(\underline{X}))] \text{ and } s \in \mathbb{Z}[\gamma(K(\underline{X}))] \right\} \cdot$$

For an ideal I of $K[\underline{X}]$, $\mathrm{Z}(I)$ will denote *the algebraic set* of K^m defined by

$$\mathrm{Z}(I) = \{x \in K^m \mid f(x) = 0 \quad \text{for all } \; f \in I\},$$

and $\mathcal{J}(\mathrm{Z}(I))$ will denote *the ideal* of $K[\underline{X}]$ associated to I, that is:

$$\mathcal{J}(\mathrm{Z}(I)) = \{f \in K[\underline{X}] \mid f(x) = 0 \quad \text{for all } \; x \in \mathrm{Z}(I)\} \cdot$$

For a prime ideal I of $K[\underline{X}]$, $k(I)$ denotes the residue field of I, that is the quotient field of the integral domain $K[\underline{X}]/I$.

La notion of p-adic ideal was introduced in [13] as a p-adic analogue of that of real ideal. It was used with the model-completeness of the theory of p-adically closed fields to give an alternative proof of the p-adic Nullsellensatz. In this paper a p-adic ideal is in fact a p-adic ideal of p-rank 1 in the sense of [13]. Let us recall the definition of this notion:

Definition 1.1. *Let I be an ideal of the polynomial ring $K[\underline{X}]$ generated by the polynomials $f_1, \ldots, f_r$. We say that I is a p-adic ideal if for every polynomial $g \in K[\underline{X}]$, for every $q \in \mathbb{N}^*$ and for every $\lambda_1, \ldots, \lambda_r \in \Lambda \cdot K[\underline{X}]$ such that $g^q = \lambda_1 f_1 + \cdots + \lambda_r f_r$ then we have $g \in I$.*

For p-adic prime ideals of $K[\underline{X}]$, we have the following characterization:

Proposition 1.2 ([13]). *Let I be a prime ideal of $K[\underline{X}]$. Then I is a p-adic ideal of $K[\underline{X}]$ if and only if the residue field $k(I)$ of I is a formally p-adic field.*

In particular, it is clear to see that the prime ideal $(X_1, \ldots, X_i)$ of $K[\underline{X}]$ generated by $X_1, \ldots, X_i$ is a p-adic ideal for every integer i such that $1 \le i \le m$. We need also the following result from [13]:

Proposition 1.3. *An ideal I of $K[\underline{X}]$ is p-adic if and only if $\mathcal{J}(\mathrm{Z}(I)) = I$.*

The first important model-theoretic result about p-adically closed fields is the model-completeness, which says that the theory of p-adically closed fields, denoted by $\mathrm{Th}(\mathbb{Q}_p)$, is model-complete in the language of rings. For $n \in \mathbb{N}^*$, let P_n (*resp.* P_n^*) denote the unary predicate interpreted as follows

$$\forall\, x \; (\, \mathrm{Th}(\mathbb{Q}_p) \models P_n(x) \;\longleftrightarrow\; \exists\, y,\, x = y^n)$$

$$\big(\textit{resp.} \,\forall\, x \,\big(\mathrm{Th}(\mathbb{Q}_p) \models P_n^*(x) \;\longleftrightarrow\; x \neq 0 \,\wedge\, P_n(x)\big)\big)\,.$$

The other important model-theoretic result is elimination of quantifiers. In this context, Macintyre's theorem [8] states that $\mathrm{Th}(\mathbb{Q}_p)$ admits elimination of quantifiers in the language of rings extended by the predicates P_n^* for $n \geq 1$. This result was generalized by in [10] to the theory of p-adically closed fields of p-rank d.

2 The p-adic spectrum

The p-adic spectrum of a ring was introduced by E. Robinson [11] in order to give a p-adic analogue of the real spectrum of a ring. An alternative definition of the p-adic spectrum was given in [5]. Here we follow that of [2]. Let A be a ring, and let us consider the relation $\sim_p$ on the homomorphisms $A \longrightarrow L$, where $L \models \mathrm{Th}(\mathbb{Q}_p)$. Let $f : A \longrightarrow L$ and $g : A \longrightarrow M$ be two homomorphisms, where L, $M \models \mathrm{Th}(\mathbb{Q}_p)$. We say that f and g are equivalent, denoted $f \sim_p g$, if there exists $\varphi : L \longrightarrow M$ such that $g = \varphi \circ f$. The model-completeness of the theory $\mathrm{Th}(\mathbb{Q}_p)$ insures that the relation $\sim_p$ is an equivalence relation. We can now state:

Definition 2.1. *Let A be a ring. The p-adic spectrum of A, denoted by $Spec_p(A)$, is the topological space defined by*

$$Spec_p(A) = \left\{ A \longrightarrow L \;\middle|\; \textit{where } L \textit{ is a } p\textit{-adically closed field} \right\} \Big/ \sim_p,$$

and whose topology is given by the open basis

$$\left\{ D_{\underline{n}}(\underline{a}) \middle| n_i \in \mathbb{N} \textit{ and } a_i \in A \textit{ for } 1 \leq i \leq r \right\}, \textit{ where}$$

$$D_{\underline{n}}(\underline{a}) = \left\{ (A \xrightarrow{\varphi} L)/\sim_p \;\middle|\; L \models \bigwedge_{i=1}^{r} P^*_{n_i}(\varphi(a_i)) \right\}.$$

Using Macintyre's theorem, it is not difficult to see that $D_{\underline{n}}(\underline{a})$ are well-defined.

Definition 2.2. *Let A be a ring. We call a constructible subset of $Spec_p(A)$ any subset which can be obtained by taking a finite number of intersections, unions and complements from open basis.*

Remark 2.3. *Let A be a ring. Then the p-adic spectrum $Spec_p(A)$ of A is a quasi-compact space. Every constructible subset is also a quasi-compact space.*

The coset of a homomorphism $\alpha : A \longrightarrow k(\alpha)$ modulo $\sim_p$, where $k(\alpha)$ is a p-adically closed field, will also denoted by α. For every a in A, we will denote

$a(\alpha) = \alpha(a)$. Let α and β be in $\mathrm{Spec}_p(A)$. We say that β is a specialization of α (or α is a generalization of β) if $\beta \in \overline{\{\alpha\}}$, that is

$$\forall\, a \in A, \quad \big(a(\beta) \in k(\beta)^{(n)} \quad \Longrightarrow \quad a(\alpha) \in k(\alpha)^{(n)}\big)\,.$$

The specialization yields a partial order on $\mathrm{Spec}_p(A)$: $\alpha \leq \beta$ if and only if β is a specialization of α. A specialization chain of length r in $\mathrm{Spec}_p(A)$ is a sequence $(\alpha_i)_{0\leq i\leq r}$ of elements of $\mathrm{Spec}_p(A)$ such that $\alpha_0 < \alpha_1 < \ldots < \alpha_r$. The dimension of a constructible subset C of $\mathrm{Spec}_p(A)$, dented by $\dim(C)$, is the maximum length of specialization chains of $\mathrm{Spec}_p(A)$ contained in C, if there exist. Otherwise, the dimension is infini.

Proposition 2.4. *Let A be a ring. Let C be a constructible set of the p-adic spectrum $Spec_p(A)$ of A. Then*

$$\overline{C} = \{\beta \in Spec_p(A) |\, \exists\, \alpha \in C \text{ such that } \beta \text{ is a specialization of } \alpha\}\cdot$$

Proof. Let $\beta \in \mathrm{Spec}_p(A)$ such that there exists $\alpha \in C$ with β is a specialization of α. Since $\beta \in \overline{\{\alpha\}}$, we have $\{\alpha\} \cap V \neq \emptyset$ for every neighborhood V of β. But $\{\alpha\} \subset C$ implies that $C \cap V \neq \emptyset$ for every neighborhood V of β. Thus, $\beta \in \overline{C}$.

Conversely, let $\beta \in \overline{C}$. For every open basis subset $D_{\underline{n}}(\underline{a})$ which contains β, the set $D_{\underline{n}}(\underline{a}) \cap C$ is not empty. Since the $D_{\underline{n}}(\underline{a})'s$ and C are quasi-compact, there is a point $\alpha \in C$ which belongs to the $D_{\underline{n}}(\underline{a})'s$. Therefore β is a specialization of α. □

Corollary 2.5. *Let A be a ring. Let C and D be two constructible sets of $Spec_p(A)$. Then C is closed (resp. open) in D if and only if it is closed with respect to specialization (resp. generalization).*

3 Main results

Let us consider the canonical injective map $K^m \longrightarrow \mathrm{Spec}_p\big(K[\underline{X}]\big)$, $x \longmapsto \bar{\varphi}_x$ defined by the equality $\varphi_x(f) = f(x)$ for all $f \in K[\underline{X}]$. We shall identify K^m with its image in $\mathrm{Spec}_p\big(K[\underline{X}]\big)$ by mean of this map. The proof of our main result is based on the following lemma which gives a characterization of the points the p-adic spectrum $\mathrm{Spec}_p\big(K[\underline{X}]\big)$ in terms of p-adic prime ideals and homomorphisms:

Lemma 3.1. *The following statements are equivalent:*

1) *There exists a point $\overline{\alpha}$ of $Spec_p\big(K[\underline{X}]\big)$.*

2) *There exists an ordered pair $(I_\alpha,\, \Psi_\alpha)$, where I_α is a p-adic prime ideal of $K[\underline{X}]$ and Ψ_α a K-homomorphism from the residue field $k(I_\alpha)$ of the ideal I_α to some p-adically closed field.*

Proof. Let $\alpha : K[\underline{X}] \longrightarrow L$ be a K-homomorphism, where L is a p-adically closed field. Put $I_\alpha = \ker(\alpha)$. It is clear that I_α is a prime ideal of $K[\underline{X}]$. According to proposition 1.3 to show that I_α is p-adic, it suffices to check that $\mathcal{J}(\mathcal{Z}(I_\alpha)) = I_\alpha$. Let $f \in \mathcal{J}(\mathcal{Z}(I_\alpha))$. Let us put $\underline{x} = (x_1, \dots, x_m)$, where for each integer i such that $1 \leq i \leq m$, one has $x_i = \alpha(X_i)$. Since $g(\underline{x}) = \alpha(g(X_1, \dots, X_m)) = \alpha(g) = 0$ for all $g \in I_\alpha$, we deduce that $\underline{x} \in \mathcal{Z}(I_\alpha)$. It follows that $f(\underline{x}) = 0$. But $f(\underline{x}) = \alpha(f)$. Therefore $\alpha(f) = 0$. Thus $f \in I_\alpha$ and $\mathcal{J}(\mathcal{Z}(I_\alpha)) = I_\alpha$. According to proposition 1.2, the residue field $k(I)$ of I is a formally p-adic field. Let $k(\alpha)$ denote the p-adic closure of $k(I)$. Then $k(\alpha)$ is a p-adically closed field. The canonical map $\Psi_\alpha : k(I_\alpha) \longrightarrow k(\alpha)$ is a K-homomorphism. Thus we have an ordered pair (I_α, Ψ_α). If $\beta : K[\underline{X}] \longrightarrow M$ is an K-homomorphism, where M is a p-adically closes field. Then there exists a K-homomorphism $i : L \longrightarrow M$ such that $\beta = i \circ \alpha$. Since

$$\forall f \in K[\underline{X}], \quad \alpha(f) = 0 \iff i(\alpha(f)) = 0 \iff \beta(f) = 0,$$

we have $I_\beta = I_\alpha$. Therefore $k(I_\beta) = k(I_\alpha)$ and $\Psi_\beta = \Psi_\alpha$. Consequently, $(I_\alpha, \Psi_\alpha) = (I_\beta, \Psi_\beta)$.

Conversely, let (I_α, Ψ_α) be an ordered pair, where I_α is a p-adic prime ideal of $K[\underline{X}]$ and Ψ_α a K-homomorphism from the residue field $k(I_\alpha)$ of I_α to some p-adically closed field $k(\alpha)$. Let $s : K[\underline{X}] \longrightarrow K[\underline{X}]/I_\alpha$ be the canonical map and let $i_\alpha : K[\underline{X}]/I_\alpha \longrightarrow k(I_\alpha)$ be the injection map. Then

$$K[\underline{X}] \xrightarrow{s} K[\underline{X}]/I_\alpha \xrightarrow{i_\alpha} k(I_\alpha) \xrightarrow{\Psi_\alpha} k(\alpha).$$

The map $\varphi = \Psi_\alpha \circ i_\alpha \circ s$ is a K-homomorphism from $K[\underline{X}]$ to $k(\alpha)$. The coset of φ modulo $\sim_p$ provides a point of the p-adic spectrum $\mathrm{Spec}_p\big(K[\underline{X}]\big)$. □

Now we can state the main result of the paper which gives a new characterization of the points of the p-adic spectrum:

Theorem 3.2 (Main result). *The following statements are equivalent:*

1 *There exists a point $\overline{\alpha}$ of $Spec_p\big(K[\underline{X}]\big)$.*

2 *There exists $\big(I_\alpha, (k(I_\alpha) \cap L^{(n)})_{n\geq 1}\big)$, where I_α is a p-adic prime ideal of $K[\underline{X}]$ and L a p-adically closed extension field of the residue field $k(I_\alpha)$ of I_α.*

Proof. According to Lemma 3.1, to give a point $\overline{\alpha}$ of $\mathrm{Spec}_p\big(K[\underline{X}]\big)$ is equivalent to give an ordered pair (I_α, Ψ_φ), where I_α is a p-adic prime ideal of $K[\underline{X}]$ and Ψ_α is a K-homomorphism from the residue field $k(I_\alpha)$ to a p-adically closed field L. Let us consider $k(I_\alpha) \cap L^{(n)}$, for every $n \geq 1$ and where L is a p-adically closed extension field of $k(I_\alpha)$. Thus, to give a point

$\overline{\alpha}$ of $\mathrm{Spec}_p\big(K[\underline{X}]\big)$ is equivalent to give an ordered pair $\big(I_\alpha,\ (k(I_\alpha)\cap L^{(n)})_{n\geq 1}\big)$, where I_α is a p-adic prime ideal of $K[\underline{X}]$ and L is a p-adically closed extension field of $k(I_\alpha)$. □

The following result is the p-adic analogue of proposition 7.5.3 of [4]:

Corollary 3.3. *There is a specialization chain $\alpha_0 < \alpha_1 < \ldots < \alpha_m$ in the p-adic spectrum $Spec_p\big(K[\underline{X}]\big)$ of length m.*

Proof. According to Proposition 1.2, the prime ideal $I_0 = (X_1, \ldots , X_m)$ is a p-adic ideal of $K[\underline{X}]$. On the other hand, we know that $K[\underline{X}]/(X_1, \ldots , X_m) \simeq K$. Then $k(I_0) \simeq K$. From Theorem 3.2, $\alpha_0 = (I_0, (K^{(n)})_{n\geq 1})$ is a point of $\mathrm{Spec}_p\big(K[\underline{X}]\big)$. Similarly, the prime ideal $I_1 = (X_2, \ldots , X_m)$ of $K[\underline{X}]$ is a p-adic ideal of $K[\underline{X}]$. We have also $K[\underline{X}]/(X_2, \ldots , X_m) \simeq K[X_1]$. Thus, $k(I_1) \simeq K(X_1)$. According to example 2.2 of [9], $K(X_1)$ is a formally p-adic field. Let $k(\alpha_1)$ denote the p-adic closure of this field. Then $\alpha_1 = (I_1, (K(X_1)\cap k(\alpha_1)^{(n)})_{n\geq 1})$ is a point of $\mathrm{Spec}_p\big(K[\underline{X}]\big)$. Moreover, one has $\alpha_0 < \alpha_1$. Indeed, let $f \in K[\underline{X}]$ such that $f(\alpha_1) \in k(\alpha_1)^{(n)}$. i.e. $f(X_1, 0, \ldots , 0) \in k(\alpha_1)^{(n)}$. But $f(\alpha_0) = f(0, 0, \ldots , 0)$. Therefore $f(\alpha_0) \in k(\alpha_0)^{(n)}$. Since $\alpha_0 \notin \overline{\{\alpha_1\}}$, one has $\alpha_0 < \alpha_1$.

Let $\alpha_i = (I_i, (k(I_i)\cap k(\alpha_i)^{(n)})_{n\geq 1})$, with $I_i = (X_{i+1}, \ldots , X_m)$, $k(I_i)$ the residue field of I_i and $k(\alpha_i)$ the p-adic closure of $k(I_i)$ for $1 \leq i < m$. We have to construct the point α_{i+1}. As above, the prime ideal $I_{i+1} = (X_{i+2}, \ldots , X_m)$ is a p-adic ideal of $K[\underline{X}]$. But $K[\underline{X}]/I_{i+1} \simeq K[X_1, \ldots , X_{i+1}]$. Therefore $k(I_{i+1}) \simeq k(I_i)(X_{i+1})$. Thus $k(I_{i+1})$ is a formally p-adic field over $k(I_i)$. Let $k(\alpha_{i+1})$ denote the p-adic closure of $k(I_{i+1})$. Put $\alpha_{i+1} = (I_{i+1}, (k(I_{i+1})\cap k(\alpha_{i+1})^{(n)})_{n\geq 1})$. Then α_{i+1} is a point of $\mathrm{Spec}_p\big(K[\underline{X}]\big)$ and $\alpha_i < \alpha_{i+1}$. We put
$\alpha_m = ((0), (K(\underline{X})\cap k(\alpha_m)^{(n)})_{n\geq 1})$. □

4 Applications to semi-algebraic sets

Let us recall that a *p-adic semi-algebraic set* of K^m is a subset which can be obtained by taking a finite number of intersections, unions and complements from subsets of the form $\big\{x \in K^m \,|\, f(x) \in K^{(n)}\big\}$, with $f \in K[\underline{X}]$. It is easy to see that any semi-algebraic set S of K^m can be written of the form

$$S = \bigcup_{j=1}^{r}\bigcap_{i=1}^{q}\left\{x \in K^m \middle|\, g_j(x) = 0 \ \text{ and } \ f_{ij}(x) \in K^{*(n_{ij})}\right\},$$

where $g_j,\ f_{ij} \in K[\underline{X}]$ and $n_{ij} \in \mathbb{N}^*$.

Let us remark that the p-adic semi-algebraic sets are the p-adic analogue of the real semi-algebraic sets, that is the subset of $\mathbb{R}^m$ which can be obtained by taking a finite number of intersections, unions and complements from subsets

of the form $\{x \in \mathbb{R}^m \mid f(x) \geq 0\}$, with $f \in \mathbb{R}[\underline{X}]$ (*see* [4]). Macintyre's theorem enables us to deduce that the projection of a semi-algebraic subsets of K^{m+1} is a semi-algebraic subsets of K^m. Thus, the closure and the interior of a p-adic semi-algebraic set are also p-adic semi-algebraic sets. Using Macintyre's theorem, we can also conclude that the p-adic semi-algebraic sets of K^m are exactly the definable subsets of K^m. We refer the reader to [3] for more details about definable subsets.
The dimension of a p-adic semi-algebraic set S of K^m, denoted $\dim(S)$, is the Krull dimension of the ring $K[\underline{X}]/\mathcal{J}(S)$, where $\mathcal{J}(S)$ is the ideal of $K[\underline{X}]$ defined by

$$\mathcal{J}(S) = \{f \in K[\underline{X}] \mid f(x) = 0 \text{ for all } x \in S\}.$$

A *p-adic semi-algebraic function* is a function $f : S \longrightarrow K$ whose graph is a p-adic semi-algebraic subset of K^{m+1}. For more details about p-adic semi-algebraic subsets and functions, we refer the reader to [7] and [12]. Here we will need the following result (*see* [12]):

Proposition 4.1 (Corollary 3.1, [12]). . *Let S be a p-adic semi-algebraic subset of K^m. Then there is a finite partition of S into p-adic semi-algebraic subsets $S_1, \ldots, S_q$, where each S_i is homeomorphic (by means of a semi-algebraic function) to a p-adic semi-algebraic open subset of K^{r_i} with $r_i \leq m$.*

We will need the following result about the dimension of p-adic semi-algebraic sets:

Proposition 4.2 (Theorem 3.2, [12]). . *Let $S = S_1 \cup \cdots \cup S_q$ be a p-adic semi-algebraic subset of K^m, where each S_i is homeomorphic (by means of a semi-algebraic function) to a p-adic semi-algebraic open subset of K^{r_i} with $r_i \leq m$. Then we have* $\dim(S) = \max(\dim(S_1), \ldots, \dim(S_r))$.

As in the real case, we can define the p-adic tilde operation as follows:

Definition 4.3. *The p-adic tilde operation is the correspondance $S \longmapsto \widetilde{S}$, where*

$$S = \bigcup_{j=1}^{r} \bigcap_{i=1}^{q} \left\{ x \in K^m \,\middle|\, g_j(x) = 0 \text{ and } f_{ij}(x) \in K^{*(n_{ij})} \right\}$$

is a p-adic semi-algebraic subset of K^m and

$$\widetilde{S} = \bigcup_{j=1}^{r} \bigcap_{i=1}^{q} \left\{ \alpha \in \mathrm{Spec}_p\big(K[\underline{X}]\big) \,\middle|\, g_j(\alpha) = 0 \text{ and } f_{ij}(\alpha) \in k(\alpha)^{*(n_{ij})} \right\}$$

is a constructible subset of $\mathrm{Spec}_p\big(K[\underline{X}]\big)$, where $f_{ij}, g_j \in K[\underline{X}]$.

The following result summarizes the main properties of the p-adic tilde operation:

Theorem 4.4. *i) The map* $S \longmapsto \widetilde{S}$ *is an isomorphism between the* p*-adic semi-algebraic subsets of* K^m *and the constructible subsets of the* p*-adic spectrum* $Spec_p\big(K[\underline{X}]\big)$.

(ii) A semi-algebraic subset S *is open in* K^m *if and only if* $\widetilde{S}$ *is open in* $Spec_p\big(K[\underline{X}]\big)$.

Thus, the isomorphism $S \longmapsto \widetilde{S}$ *yields a bijection between the* p*-adic semi-algebraic open subsets of* K^m *and the quasi-compact constructible subsets of* $Spec_p\big(K[\underline{X}]\big)$.

(iii) The p*-adic tilde operation commutes with the closure and the interior, that is, if* S *is a* p*-adic semi-algebraic subset of* K^m*, then:* $\widetilde{\overline{S}} = \overline{\widetilde{S}}$ *and* $\widetilde{\dot{S}} = \dot{\widetilde{S}}$.

Proof. i) Let S and T be two p-adic semi-algebraic subsets of K^m such that $S = T$. The model-completeness of the theory $\mathrm{Th}(\mathbb{Q}_p)$ implies that $S = T$ on L, where L is any p-adically closed fields such that $K \subset L$. In particular, we have $S = T$ on $k(\alpha)$ for any $\alpha \in \mathrm{Spec}_p\big(K[\underline{X}]\big)$. Thus, $\widetilde{S} = \widetilde{T}$. It follows that the map $S \longmapsto \widetilde{S}$ is well-defined. Conversely, if $\widetilde{S} = \widetilde{T}$. Then $\widetilde{S} \cap K^m = \widetilde{T} \cap K^m$. Therefore $S = T$. Thus, the map $S \longmapsto \widetilde{S}$ is injective. On the other hand, it is clear that this map is an homomorphism surjective from the Boole algebra of p-adic semi-algebraic subsets of K^m to that of the constructible of $\mathrm{Spec}_p\big(K[\underline{X}]\big)$.
ii) Let S be an open semi-algebraic subsets of K^m. According to the p-adic finiteness theorem, S is a finite union of open of the type

$$S_{f_1,\dots,f_r} = \left\{ x \in K^m \,\middle|\, f_1(x) \in K^{*(n_1)}, \dots, f_r(x) \in K^{*(n_r)} \right\},$$

where $f_1, \dots, f_r \in K[\underline{X}]$. Thus, $\widetilde{S}$ is a finite union of open of the type $\widetilde{S_{f_1,\dots,f_r}}$.
iii) Since quasi-compacts open form an open basis of the space $\mathrm{Spec}_p\big(K[\underline{X}]\big)$, we have $\widetilde{\dot{S}} = \bigcup\limits_{\widetilde{O} \subset \widetilde{S}} \widetilde{O} = \bigcup\limits_{O \subset S} O$, where O are open semi-algebraic contained in S. But $\dot{S}$ is a p-adic semi-algebraic. Therefore $\widetilde{\dot{S}} = \dot{\widetilde{S}}$. Similarly for the closure.□

The following result is the p-adic analogue of proposition 7.5.6 of [4]:

Proposition 4.5. *Let* S *be a semi-algebraic of* K^m*. Then* $\dim(S) = \dim(\widetilde{S})$.

Proof. Let $\alpha_r < \dots < \alpha_0$ be a specialization chain of length r of $\mathrm{Spec}_p\big(K[\underline{X}]\big)$ contained in $\widetilde{S}$. Let us put $I_i = \ker(\alpha_i)$ for each integer i such that $0 \le i \le r$. Then, we have a chain

$$I_0 \subset I_1 \subset \dots \subset I_r$$

of prime ideals of $K[\underline{X}]$ of length r. On the other hand, if $\alpha \in \widetilde{S}$ then we have $f(\alpha) = 0$ for all $f \in \mathcal{J}(S)$. It follows that $\mathcal{J}(S) \subset I_i$ for each integer i such that $0 \le i \le r$. Therefore we we have a chain

$$I_0/\mathcal{J}(S) \subset I_1/\mathcal{J}(S) \subset \dots \subset I_r/\mathcal{J}(S)$$

of prime ideals of $K[\underline{X}]/\mathcal{J}(S)$ of length r. Hence $\dim(\widetilde{S}) \leq \dim(S)$.

Conversely, we have to show that $\dim(S) \leq \dim(\widetilde{S})$. Let $d = \dim(S)$. According to Proposition 4.2, we know that S contains a p-adic semi-algebraic subset T such that T is homeomorphic (by means of a semi-algebraic function) to a p-adic semi-algebraic open subset U of K^d which contains the origin. The p-adic semi-algebraic homeomorphism between T and U extends to a homeomorphism between $\widetilde{T}$ and $\widetilde{U}$. From Corollary 3.3, there exists in $\mathrm{Spec}_p\big(K[X_1, \dots, X_d]\big)$ a specialization chain of length d which ends by the origin. According to corollary 2.4, $\widetilde{U}$ is closed with respect to generalization, and this specialization chain is contained in $\widetilde{U}$. Thus, we obtain a specialization chain of length d contained in $\widetilde{T}$, and also in $\widetilde{S}$. Hence $\dim(S) \leq \dim(\widetilde{S})$. □

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