International Journal of Algebra, Vol. 2, 2008, no. 20, 995 - 998

When an Irreducible Submodule is Primary

A. Yousefian Darani

Department of Mathematics
University of Mohaghegh Ardabili
P.O. Box 179, Ardabil, Iran
yousefian@uma.ac.ir

Abstract

Let R be a commutative ring, M an R-module and N an irreducible submodule of M. In this paper we provide a necessary and sufficient condition under which N is primary.

Mathematics Subject Classification: 13C05, 13C13

Keywords: Primal submodule, Irreducible submodule, Primary submodule

1 Introduction

Throughout this paper R will denote a commutative ring with a nonzero identity and M is an R-module. To extend the concepts of prime and primary ideal from the category of rings to the category of modules has stimulated several authors to show that many, but not all, of the results in the theory of rings are also valid for modules. In this short note, we will prove a result about primary submodules which has proved for primary ideals in [3].

It is well-known that if M is a Noetherian R-module, then every irreducible submodule of M is primary (see for example [4]). This is an important fact in the theory of submodules, since it admits the proof of the existence of a primary decomposition for an arbitrary submodule of M. one interesting and important question is the following:

What is the necessary and sufficient condition under which an irreducible submodule of M is primary?

Our aim in this paper is to prepare such a condition. We will prove (Theorem 3.3) that if N is an irreducible submodule of M, then N is primary if and only if N satisfies the principal quotient-chain condition.

2 Preliminary Notes

In this section, we provide the definitions and notations that we will use throughout.

If N is a submodule M, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N :_R M)$. Then $(0 :_R M)$ is the annihilator of M, $ann(M)$.

Definition 2.1 *Let N be a submodule of M, $a \in R$ and let $f_a : M/N \to M/N$ be the canonical homomorphism produced by multiplication by a. N is called a primary submodule of M if N is proper and, for every $a \in R$, f_a is either injective or nilpotent. So N is primary if and only if whenever $am \in N$, for some $a \in R$, $m \in M$, then either $m \in N$ or $a^n M \subseteq N$ for some positive integer n. If N is primary, then $P := Rad(N :_R M)$ is a prime ideal of R and we say that N is P-primary.*

Definition 2.2 *Let N be a submodule of M. An element $a \in R$ is called prime to N if $am \in N$ $(m \in M)$ implies that $m \in N$. In this case $(N :_M a) = \{m \in M : am \in N\} = N$. Denote by $S(N)$ the set of all elements of R that are not prime to N. A proper submodule N of M is said to be primal if $S(N)$ forms an ideal of R; this ideal is called the adjoint ideal P of N. In this case we also say that N is a P-primal submodule ([1], [2]).*

Lemma 2.3 *If N is a P-primary submodule of M, then N is P-primal.*

Proof. Assume that $a \in P$. Let n be the smallest positive integer for which $a^n \in (N :_R M)$. As N is proper, $a(a^{n-1}m) = a^n m \in N$ for some $m \in M \setminus N$. This implies that $a \in S(N)$. If $r \in R \setminus P$ and $rm \in N$, then $m \in N$. So r is prime to N, that is $r \notin S(N)$. We have already shown that P consists of those elements of R that are not prime to N. So N is P-primal.

Definition 2.4 *N is called an irreducible submodule if N can not be expressed as a finite intersection of proper divisors of N.*

3 Main Result

We start by the following definition:

Definition 3.1 *(QUOTIENT-CHAIN CONDITION) Let N be a submodule of M. Every quotient-chain of submodules,*

$$N \subset N_1 \subset N_2 \subset ... \subset N_i \subset N_{i+1} \subset ...$$

where N_i is of the form $(N_{i-1} :_M I_i)$ and is a proper over-submodule of N_{i-1} for some ideal I_i of R, breaks of after a finite number of terms.

Definition 3.2 *Let N be a submodule of M. A chain*

$$N \subset N_1 \subset N_2 \subset ... \subset N_i \subset N_{i+1} \subset ...$$

is said to be a principal quotient-chain *if there is some $b \in R$ such that $N_1 = (N :_M Rb)$ and $N_k = (N_{k-1} :_M Rb) = (N :_M Rb^n)$ for every positive integer k. We say that for a submodule N of M, the principal quotient-chain condition holds, if every principal quotient-chain with the first term N terminates. Now we imply the main theorem.*

Theorem 3.3 *Let N be an irreducible submodule of M. Then N is primary if and only if N satisfies the principal quotient-chain condition.*

Proof. First Assume that N is P-primary, and assume that

$$N \subset N_1 \subset N_2 \subset ... \subset N_i \subset N_{i+1} \subset ...$$

is a principal quotient-chain, where $N_1 = (N :_M Rb)$ and $N_k = (N_{k-1} :_M Rb) = (N :_M Rb^n)$ for some $b \in R$ and for every positive integer k. If $b \in P$, then $b^n \in (N :_R M)$ for some positive integer n, so that $(N :_M Rb^n) = (N :_M Rb^{n+i}) = M$ for every positive integer i. So assume that $b \in R \backslash P$. By Lemma 2.3, N is P-primal. So b is prime to N. It follows that $N = (N :_M Rb) = (N :_M Rb^n)$ for every positive integer n. Therefore the principal quotient-chain condition holds for N. Conversely, assume that N satisfies the principal quotient-chain condition. pick an element $m \in M \backslash N$ and an element $b \in R \backslash Rad(N :_R M)$. By hypothesis, the chain of the submodule quotients

$$N \subset (N :_M Rb) \subset (N :_M Rb^2) \subset ... \subset (N :_M Rb^i) \subset (N :_M Rb^{i+1}) \subset ...$$

must be stationary. Thus $(N :_M Rb^n) = (N :_M Rb^{n+1})$ for a positive integer n. Clearly the submodules $K := N + Rm$ and $L := N + Mb^n$ are proper divisors of N. As N is irreducible, there exists an element $l \in K \cap L$ with $l \notin N$. Thus $l = n + rm = n' + m'b^n$ for some $n, n' \in N$, $m' \in M$ and $r \in R$. If $bm \in N$, then $bn' + m'b^{n+1} = bl = bn + brm \in N$ implies that $m'b^{n+1} \in N$. This show that $m' \in (N :_M Rb^{n+1}) = (N :_M Rb^n)$. So from $m'b^n \in N$ we get $l \in N$, which is a contradiction. So $bm \notin N$ shows that N is primary.

Corollary 3.4 *Every irreducible submodule of M is primary if and only if every irreducible submodule of M satisfies the principal quotient-chain condition.*

References

[1] J. Dauns, Primal modules, Comm. Algebra, **25(8)** (1997), 2409-2435.

[2] S. Ebrahimi Atani and A. Yousefian Darani, Some remarks on primal submodules, Sarajevo Journal of Mathematics, to appear.

[3] L. Fuchs, A condition under which an irreducible ideal is primary, Quart. Journ. of Math., **19**(1948), 235-237.

[4] R. Y. Sharp, Steps in Commutative Algebra, Cambridge University Press, Cambridge, 1990.

Received: October 1, 2008