Invariant and Reducing Subspaces
of Composition Operators

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I Preliminaries :Let \((X, s, \mu)\) be a \(\sigma\)- finite measure space and let \(\varphi : X \to X\) be a non-singular measurable transformation. Then a composition transformation \(C_\varphi : L^p(X, s, \mu) \to L^p(X, s, \mu)\) is defined by

\[ C_\varphi(f) = f \circ \varphi \text{ for all } f \in L^p(X, s, \mu) \]

If \(C_\varphi\) is continuous, we shall call \(C_\varphi\) a composition operator induced by \(\varphi\). If \(X = \mathbb{N}\), the set of natural numbers and \(\mu\) is the counting measure on \(\mathcal{P}(\mathbb{N})\), the power set of \(\mathbb{N}\), then \(L^p(X, s, \mu) = \ell^p\), the Banach space of square summable sequence of complex numbers. For \(p = 2\), \(\ell^2\) is the Hilbert space. This is the earliest known example of Hilbert space found by Hilbert himself in 1912. The set \(\{e_n : n \in \mathbb{N}\}\) where \(e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)\) is a basis for \(\ell^2\).

Let \(T : \mathbb{N} \to \mathbb{N}\) be a mapping. Two positive integers \(m\) and \(n\) are said to be in the same orbit of \(T\) if there exist two positive integers \(r\) and \(s\) such that \(T^r(m) = T^s(n)\).

Here and elsewhere, \(T^r\) denotes the composition of \(T\) with itself \(r\) times. If \(n \in \mathbb{N}\), then \(O_T(n) = \{m \in \mathbb{N} : T^r(m) = T^s(n) \text{ for some } r, s \in \mathbb{N}\}\) is called the orbit of \(n\) with respect to \(T\).

A mapping \(T : \mathbb{N} \to \mathbb{N}\) is said to be antiperiodic at \(n\), if \(T^m(n) \neq n\) for every \(m \in \mathbb{N}\). If \(T\) is antiperiodic at every \(n \in \mathbb{N}\), then we say that \(T\) is purely antiperiodic. For an example, the mapping \(T : \mathbb{N} \to \mathbb{N}\) defined by

\[ T(n) = \begin{cases} n + 2, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd} \end{cases} \]

is antiperiodic at every even natural number but not antiperiodic at an odd natural number. If for a natural number \(n\) there exists \(m \in \mathbb{N}\) such that
$T^n(n) = n$, then $T$ is called periodic at $n$. If $T$ is periodic at every $n \in N$, we say that $T$ is purely periodic. For sake of convenience, we shall use $\sharp(E)$ to denote the cardinality of the set $E$.

The composition operators have been the subject matter of a systematic study over the past four decades (eg see monographs Cowen and Maccluer [4], Shapiro [17] and Singh and Summer [20]). For more information about composition operators we refer to Ridge [15], Shapiro [17], Schwartz [16], Enflo [5], Cowen [2,3,4], Matache [10,11], Nordgren [12, 13], Singh and Komal [18, 19], Carlson [1], Halmos [6,7]. One of the outstanding problems of the operator theory is the Invariant subspace problem. The problem is simple to state: “Does every bounded linear operator on a separable infinite dimension Hilbert space have a non-trivial Invariant subspace ?

Recently, Enflo [5] and Read [14] settled the problem for Banach space whereas Lomonosov solved the problem for compact operators. Singh and Komal [18, 19] has shown that every composition operator on $\ell^2$ has an invariant subspace. In this paper our study centres around the Invariant and Reducing subspaces of composition operators mainly on the Hilbert space $\ell^2$.

1. Invariant subspaces of composition operators on $L^p$ spaces

**Theorem 1.1**: If $\varphi^{-1}(E) \subseteq E$ for a measurable subset of $E$ of $X$ such that $0 < \mu(E) < \mu(X)$. Then $M = L^p(E) = \{ f \in L^p(\mu) : f(x) = 0$ a.e for all $x \not\in E \}$ is an invariant subspace of $C_\varphi$.

**Proof**: Let $M = L^p(E)$. Suppose $f \in M$. Then $f(x) = 0$ for all $x \not\in E$. Then for $x \not\in E$, $(C_\varphi f)(x) = f(\varphi(x)) = 0$ as $\varphi(x) \not\in E$ because if $\varphi(x) \in E$. Then $x \in \varphi^{-1}(E) \subseteq E$ which is a contradiction.

**Corollary 1.2**: If $\varphi^{-1}(E) = E$, then $C_\varphi$ has a reducing subspace.

**Example 1.3**: Let $X = [0, 1]$ and $\mu =$ Lebesgue measure.

Define $\varphi : [0, 1] \to [0, 1]$ by

$$\varphi(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ 2, & \frac{1}{2} < x \leq 1 \end{cases}$$

Then $\varphi^{-1}([0, \frac{1}{2}]) = [0, \frac{1}{2}]$.

Take $E = [0, \frac{1}{2}]$ and

$M = \{ f \cdot \chi_E :$ where $f \in L^p[0, 1] \}$

Now, if $g \in M$, then $g = f \cdot \chi_E$

so that
\[ C_\varphi g = C_\varphi f \cdot \chi_{\varphi^{-1}(E)} = C_\varphi f \cdot \chi_E \text{ because } \varphi^{-1}(E) = E \]

Hence \( C_\varphi g \in L^p[0, 1] \)
Thus \( M \) is a non-trivial proper invariant subspace of \( C_\varphi \)

2. Reducing subspaces of composition operators on \( \ell^2 \):

**Theorem 2.1:** Let \( \varphi : N \to N \) be an injection which is not a surjection. Then \( C_\varphi \) has a reducing subspace if and only if there exists two points in \( N \) which are not in the same orbit of \( \varphi \).

**Proof:** We first assume that there are two points \( m_0 \) and \( n_0 \in N \) which are not in the same orbit of \( \varphi \). Then \( O_\varphi(n_0) \cap O_\varphi(m_0) = \phi \).

Take \( M = \ell^2(N) \cdot \chi_{O_\varphi(n_0)} \), where \( \chi_E \) denotes the characteristic function of the set \( E \). Then clearly for \( n \in O_\varphi(n_0) \) we have \( C_\varphi^* e_n = e_{\varphi(n)} \) as \( C_\varphi e_n = e_{\varphi^{-1}(n)} \) belong to \( M \) and

\[ M = \text{span}\{C_\varphi^m(C_\varphi^n) : m, n \in N \cup \{0\}\} \]

Hence, \( M \) is a reducing subspace of \( C_\varphi \).

Conversely, suppose that \( M \) is a reducing subspace of \( C_\varphi \). We prove that there exist two points in \( N \) which are not in the same orbit of \( T \).

Let \( E = \{n \in N : \sharp(T^{-1}(n)) = 0\} \). Since \( T \) is not surjective, so \( \sharp(E) \geq 1 \).

Let \( P \) be the projection on \( M \). If \( \sharp(E) = 1 \), then for \( n \in N \), we have

\[ PC_\varphi e_n = C_\varphi Pe_n = 0, \]

which implies that \( Pe_n \in \ker C_\varphi \) so that \( Pe_n = \alpha e_n \).

Hence \( e_n \in M \). Therefore \( \{(C_\varphi^*)^m(C_\varphi)^n e_n : n \in N \cup \{0\}\} \subseteq M \) so that \( \ell^2(N) \chi_{O_\varphi(n)} \subseteq M \).

Next, if \( O_\varphi(n) = N \), then \( M = \ell^2(N) \), which is a contradiction.

Hence, \( \sharp(E) \geq 2 \). Let \( m_o, n_o \in E \). Clearly \( O_\varphi(m_o) \cap O_\varphi(n_o) = \phi \).

Therefore \( m_o \) and \( n_o \) are not in the same orbit of \( \varphi \).

**Theorem 2.2:** Let \( \varphi : N \to N \) be an injection which is not a surjection and \( \sharp(N|\varphi(N)) \geq 2 \). Then \( C_\varphi \) has at least \( \sharp(N|\varphi(N)) \) reducing subspaces of
composition operator of the type

\[ M_k = \ell^2(N)\chi_{O_{\varphi(k)}} \quad \text{for } k \in N|\varphi(N) \]

**Proof:** Suppose \( k_1, k_2 \in N|\varphi(N) \) be such that \( k_1 \neq k_2 \). Then we first show that \( O_{\varphi(k_1)} \neq O_{\varphi(k_2)} \). For if \( m \in O_{\varphi(k_1)} \cap O_{\varphi(k_2)} \), then \( \varphi^m(k_1) = m = \varphi^t(k_2) \) for some natural numbers \( s \) and \( t \).

Suppose \( t \leq s \). Then \( \varphi^{s-t}(k_1) = k_2 \) which implies that \( k_2 \in \varphi(N) \). This is a contradiction. Hence \( \varphi^s(k_1) \neq \varphi^t(k_2) \).

Similar argument holds when \( t > s \) in which case \( k_1 \in \varphi(N) \) is a contradiction.

Taking \( M_k = \text{span}\{e_n : n \in O_{\varphi(k)}\} \) where \( k \in N|\varphi(N) \) we find that, \( M_k = \{C_{\varphi}^q \text{e}_k : p, q \in N\} \), clearly \( M_k = \ell^2(O_{\varphi(k)}) = \ell^2(N)\chi_{O_{\varphi(k)}} \) and hence \( M_k \) is a reducing subspace of \( C_{\varphi} \).

Therefore we conclude that \( C_{\varphi} \) has at least \( \sharp(N|\varphi(N)) \) reducing subspaces.

**Example 2.3:** Let \( \varphi : N \to N \) be defined by \( \varphi(n) = 2n \) so that \( \sharp(N|\varphi(N)) = \infty \). Then \( C_{\varphi} \) has infinite number of reducing subspaces of the type \( M_k = \ell^2(N)\chi_{O_{\varphi(k)}} \) where \( k \in N|\varphi(N) \).

Let \( k \) be an odd natural number. Then \( k \not\in \varphi(N) \). Take \( O_{\varphi(k)} = \{k, 2k, 4k, \ldots, 2^n k, \ldots\} \).

Now \( \ell^2_{O_{\varphi(k)}} = \ell^2(N)\chi_{O_{\varphi(k)}} \) is reducing subspace of \( C_{\varphi} \). For an illustration, take \( k = 1 \), Then \( \ell^2_{\varphi(1)} = \{f \in \ell^2(N) : f(2m + 1) = 0 \text{ for } m \in N\} \)

**Theorem 2.4:** Let \( \varphi : N \to N \) be defined by \( \varphi(n) = pn + q \), where \( p, q \) are positive integers and \( q \geq 2 \). Then for each \( k, 1 \leq k \leq p + q - 1 \), \( C_{\varphi} \) has at least \( k \) reducing subspaces of the type \( M_k = \ell^2(N)\chi_{O_{\varphi(k)}} \).

**Proof:** Let \( E_{k_1} = O_{\varphi(k_1)}, E_{k_2} = O_{\varphi(k_2)} \) where \( 1 \leq k_1 \leq p + q - 1, 1 \leq k_2 \leq p + q - 1 \), and \( k_1 \neq k_2 \).

We first see that \( E_{k_1} \cap E_{k_2} = \phi \).

For if \( m \in E_{k_1} \cap E_{k_2} \), then we can write

\[ \varphi^{(s)}(k_1) = m = \varphi^{(t)}(k_2) \]

for some natural number \( s \) and \( t \) respectively.

First suppose \( s \geq t \). Then \( \varphi^{(s-t)}(k_1) = k_2 \) and

\[ p^sk_1 + p^{s-1}q + \ldots + p^{t-1}q + p^tq + \ldots + q = p^tk_2 + p^{t-1}q + \ldots + q \]

implies that

\[ p^sk_1 + p^{s-1}q + \ldots + p^{t-1}q + p^tq = p^tk_2 \]

or

\[ p^{s-t}k_1 + p^{s-1-t}q + \ldots + p^tq + q = k_2 \]
or \( k_2 > p + q \), which is not true as \( k_2 \leq p + q - 1 \). If \( s = t \), then

\[
p^s k_1 + p^{s-1} q + \ldots + pq + q = p^t k_2 + p^{t-1} q + \ldots + pq + q
\]
or

\[
p^s k_1 + p^{s-1} q + \ldots + pq + q = p^s k_2 + p^{s-1} q + \ldots + pq + q
\]

\[p^s k_1 = p^s k_2 \Rightarrow k_1 = k_2\]

which is again a contradiction. Hence \( E_{k_1} \cap E_{k_2} = \emptyset \).

Take \( M_{k_1} = \text{span}\{e_n : n \in E_{k_1}\} \). Then clearly \( M_{k_1} = \text{span}\{C^{\nu m} C^{\nu} e_n : n \in E_{k_1}\} \) so that \( M_{k_1} \) is a reducing subspace of \( C_{\varphi} \). This is true for every \( 0 \leq k_1 \leq p + q - 1 \). Thus \( C_{\varphi} \) has atleast \( p + q - 1 \) reducing subspaces of the type \( \chi_{E_{k_1}} l^2(N) = M_{k_1} \).

**Remark:** If \( z(N|\varphi(n)) = 1 \), and \( \varphi \) is purely antiperiodic, then \( C_{\varphi} \) is not reducible.

### 3. Reducing subspaces of composition operators induced by periodic mappings.

**Lemma 3.1:** Let \( \varphi : N \rightarrow N \) be periodic at \( n \in N \). Then \( M_n \) is a reducing subspace of \( C_{\varphi(n)} \) where \( C_{\varphi(n)} = C_{\varphi}/(l^2(\varphi(n))) \) if and only if

\[M_n = \{ f \in l^2(N) : \sum_{i \in O_{\varphi(n)}} f_i = 0 \} \text{ or } f = \text{span}\{\chi_{O_{\varphi(n)}}\}.
\]

**Proof:** Clearly, \( M_n \) is a reducing subspace of \( C_{\varphi(n)} \). Because if \( f = \{f_{n_1}, f_{n_2}, \ldots, f_{n_k}\} \in l^2_{O_{\varphi(n)}} \) such that \( f_{n_1} + f_{n_2} + \ldots + f_{n_k} = 0 \), then \( C_{\varphi(n)} f = \{f_{n_2}, f_{n_3}, \ldots, f_{n-1}, f_{n_1}\} \) and so \( f_{n_2} + f_{n_3} + \ldots + f_{n_k} + f_{n_1} = 0 \). Hence \( C_{\varphi(n)} f \in M_n \).

Similarly, \( C_{\varphi(n)}^* f = \{f_{n_k}, f_{n_1}, \ldots, f_{n_k-1}\} \) so that \( C_{\varphi(n)}^* f \in M_n \) as \( f_{n_k} + f_{n_1} + \ldots + f_{n_k-1} = 0 \).

Conversely, suppose that \( M_n \) is a reducing subspace of \( C_{\varphi(n)} \). Then for \( f \in M_n \),

\[
f + C_{\varphi} f + C_{\varphi_2} f + \ldots + C_{\varphi_{k-1}} f \in M \text{ or that}
\]

\[
\{f_{n_1} + f_{n_2} + \ldots + f_{n_k}, f_{n_1} + f_{n_2} + \ldots + f_{n_k}, \ldots, f_{n_1} + f_{n_2} + \ldots + f_{n_k}\}
\]

\[
= (f_{n_1} + f_{n_2} + \ldots + f_{n_k})(e_{n_1} + e_{n_2} + \ldots + e_{n_k}) \in M
\]

Hence either \( f_{n_1} + f_{n_2} + \ldots + f_{n_k} = 0 \)

or \( e_{n_1} + e_{n_2} + \ldots + e_{n_k} = 0 \).

This proves that if \( M_n \) is a reducing subspace of \( C_{\varphi(n)} \), then either

\[M_n = \{(f_{n_1}, \ldots, f_{n_k}) : \sum_{i=1}^k f_{n_i} = 0\} \text{ or } M_n = \text{Span}\{(e_{n_1} + e_{n_2} + \ldots + e_{n_k})\}.
\]
Theorem 3.2: If $\varphi : N \to N$ is periodic for every $n \in N$, then every reducing space of $C_\varphi$ is of the type $M_n = \{ f \in \ell^2(O_{\varphi(n)}) : \sum_{k \in O_{\varphi(n)}} f(k) = 0 \}$ or $M_n = \text{span}\{ \sum_{k \in O_{\varphi(n)}} e_k \}$ i.e. $M_n = \ell^2(O_{\varphi(n)})$ or of the type $\sum_{k=1}^n \oplus M_k$.

Proof: We have $N = \bigcup_{n_k \in N} O_{\varphi(n_k)}$.

Therefore $\ell^2(N) = \bigoplus_{k=1}^\infty \ell^2(O_{\varphi(n_k)})$.

Clearly $\ell^2(O_{\varphi(n_k)})$ is invariant under $C_\varphi$ for every $k = 1, 2, ...$

Again if $f = \alpha \chi_{O_{\varphi(n_k)}}$, then $C_\varphi f = \alpha \chi_{O_{\varphi(n_k)}}$ and $C_\varphi^* f = \beta \chi_{O_{\varphi(n_k)}}$ for some scalers $\alpha$ and $\beta$, so that $\text{span}\chi_{O_{\varphi(n_k)}}$ is a reducing subspace of $C_\varphi$.

Next, if $M = \ell^2(O_{\varphi(n_k)})$, it follows from the lemma, that $M$ is a reducing subspace of $C_\varphi$.

Example 3.3: Let $\varphi : N \to N$ be defined by

$$\varphi(n) = \begin{cases} n + 1, & \text{if } n \text{ is odd} \\ n - 1, & \text{if } n \text{ is even} \end{cases}$$

Then $\varphi$ is periodic at every $n \in N$ of period 2.

Let $n$ be an odd natural number. Take $E_n = \{n, n + 1\}$ and $M_n = \text{span}\{e_n + e_{n+1}\}$

Then $M_n$ is a reducing subsapce of $C_\varphi$. Further $M'_n = \text{span}\{e_1 - e_2\}$ is also a reducing subspace of $C_\varphi$. Hence the only reducing subspace of $C_\varphi$ are of the type, $\sum_{k=1}^n \oplus M_k \oplus \sum_{k=1}^m \oplus M'_k$ for every integers $n, m = 0, 1, 3, 5, 7,...$

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