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CR-Submanifolds of a Lorentzian Para-Sasakian Manifold Endowed with the Canonical Semi-Symmetric Semi-Metric Connection

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Abstract

We define the canonical semi-symmetric semi-metric connection in a Lorentzian para-Sasakian manifold and study CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with the canonical semi-symmetric semi-metric connection. Moreover, we also obtain integrability conditions of the distributions on CR-submanifolds.

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1. Introduction

The notion of CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu in [2]. Later, CR-submanifolds of Sasakian manifold were studied by M. Kobayashi in [13]. K. Motsumoto introduced the idea of Lorentzian para-Sasakian structure and studied its several properties in [7]. U. C. De and Anup Kumar Sengupta studied CR-submanifolds of a Lorentzian para-Sasakian manifold in [14]. Lovejoy S. K. Das and the author studied CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with a quarter symmetric non-metric connection in [8]. In this paper we study CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with the canonical semi-symmetric semi-metric connection.

Let ∇ be a linear connection in an n-dimensional differentiable manifold $\bar{M}$. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. In ([3], [5]) A. Friedmann and J.A. Schouten introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form. In [4], [6], [8], [9], [10], [11] and [12], some properties of semi-symmetric or quarter symmetric connection are studied.

In this paper we study CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with the canonical semi-symmetric semi-metric connection. We consider integrability conditions of distributions on CR-submanifolds with the canonical semi-symmetric semi-metric connection.

This paper is organized as follows : In section 2, we give a brief introduction of Lorentzian para-Sasakian manifold. In section 3, we study CR-submanifolds of an LP-Sasakian manifold with the canonical semi-symmetric semi-metric connection. In section 4, we discuss the integrability conditions of distributions on CR-submanifolds.

2. Lorentzian para-Sasakian manifold

Let $\bar{M}$ be an n-dimensional almost contact metric manifold with almost contact metric stucture (ϕ, η, ξ, g) such that

$$\phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \xi) = \eta(X) \tag{2.3}$$

$$g(\phi X, Y) = g(X, \phi Y) = \psi(X,Y) \tag{2.4}$$

for all vector fields X, Y tangent to $\bar{M}$. Then the structure (ϕ, η, ξ, g) is termed as Lorentzian para-contact structure [7].

Also in a Lorentzian para-contact structure the following relations hold:

$$\phi\xi = 0,\ \eta(\phi X) = 0, \text{rank}(\phi) = n - 1.$$

A Lorentzian para-contact manifold $\bar{M}$ is called Lorentzian para-Sasakian (LP-Sasakian) manifold if [14]

$$(\bar{\bar{\nabla}}_X \phi)(Y) = g(X,Y) + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.5}$$

$$\bar{\bar{\nabla}}_X \xi = \phi X \tag{2.6}$$

for all vector fields X, Y tangent to $\bar{M}$, where $\bar{\bar{\nabla}}$ is the Riemannian connection with respect to g.

3. CR-submanifolds of LP-Sasakian manifold

Definition : [1] An m-dimensional Riemannian submanifold M of a Lorentzian para-Sasakian manifold $\bar{M}$ is called a CR-submanifold if ξ is tangent to M and there exists on M a pair of differentiable distributions $(D, D^{\perp})$ such that
(i) TM orthogonally decomposes as $D \oplus D^{\perp}$,
(ii) the distribution D_x is invariant under ϕ that is $\phi D_x \subset D_x$ for each $x \in M$,
(iii) the distribution $D^{\perp}$ is ant-invariant that is $\phi D_x^{\perp} \subset T_x^{\perp} M$, where $T_x M$ and $T_x^{\perp} M$ are tangent and normal spaces of M at $x \in M$.

The distribution D(resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. The pair $(D, D^{\perp})$ is called ξ-horizontal (resp. ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x^{\perp}$) for $x \in M$.

Any vector X tangent to M is given by

$$X = PX + QX, \tag{3.1}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. For any vector field N normal to M, we put

$$\phi N = BN + CN, \tag{3.2}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN.

We remark that owing to the existence of the $1-$form η, we can define the canonical semi-symmetric semi-metric connection $\bar{\nabla}$ in any almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$ by [10]

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y - \eta(X)Y + g(X,Y)\xi \tag{3.3}$$

such that $\bar{\nabla}_X g = 2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y)$ for any $X, Y, Z \in TM$. In particular, if $\bar{M}$ is an LP-Sasakian manifold then from (2.5), (2.6) and (3.3) we have

$$(\bar{\nabla}_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi + g(X, \phi Y)\xi, \tag{3.4}$$

$$\bar{\nabla}_X \xi = \phi X. \tag{3.5}$$

We denote by g the metric tensor of $\bar{M}$ as well as that induced on M. Let $\bar{\nabla}$ be the the canonical semi-symmetric semi-metric connection on $\bar{M}$ and ∇ be the induced connection on M with respect to unit normal N.Then the gauss and Weingarten formulae for the canonical semi-symmetric semi-metric connection are given by [12]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y). \tag{3.11}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N - \eta(X)N. \tag{2.12}$$

for any $X, Y \in TM$.

4. Integrability of distributions on CR-submanifolds

Lemma 4.1. Let M be a CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then

$$P\nabla_X \phi PY - PA_{\phi QY}X = g(X, Y)P\xi + g(X, \phi Y)P\xi + \eta(Y)PX + 2\eta(X)\eta(Y)P\xi + \phi P\nabla_X Y \tag{4.1}$$

$$Q\nabla_X \phi PY - QA_{\phi QY}X = g(X, Y)Q\xi + g(X, \phi Y)Q\xi + \eta(Y)QX + 2\eta(X)\eta(Y)Q\xi + Bh(X, Y) \tag{4.2}$$

$$h(X, \phi PY) + \nabla^{\perp}_X \phi QY = \phi Q\nabla_X Y - \eta(X)\phi QY + Ch(X, Y) \tag{4.3}$$

for all $X, Y \in TM$.

Proof : By virtue of (3.1), (3.2), (3.4), (3.6) and (3.7), we get

$$\begin{aligned}
&P\nabla_X \phi PY + Q\nabla_X \phi PY + h(X, \phi PY) - PA_{\phi QY}X - QA_{\phi QY}X \\
&+\nabla^{\perp}_X \phi QY - \eta(X)\phi QY = g(X, Y)P\xi + g(X, Y)Q\xi + \eta(Y)PX \\
&+\eta(Y)QX + 2\eta(X)\eta(Y)P\xi + 2\eta(X)\eta(Y)Q\xi + g(X, \phi Y)P\xi \\
&+g(X, \phi Y)Q\xi + \phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X, Y) + Ch(X, Y).
\end{aligned}$$

Equations (4.1)-(4.3) follows by comparing the horizontal, vertical and normal components.

Lemma 4.2. Let M be a $\xi-$vertical CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then

$$\phi P[Y, Z] = A_{\phi Y}Z - A_{\phi Z}Y + \eta(Y)Z - \eta(Z)Y$$

and

$$\eta(Z)\phi Y - \eta(Y)\phi Z = 0$$

for all $Y, Z \in D^{\perp}$.

Proof : By covariant differentiation of ϕZ with respect to Y and using (3.4), (3.11) and (3.12), we have

$$\phi P\nabla_Y Z = -A_{\phi Z}Y - g(Y,Z)\xi - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi - g(Y,\phi Z)\xi - Bh(Y,Z) - \eta(Y)\phi Z$$

for all $Y, Z \in D^{\perp}$. Interchanging Y, Z and subtracting, we get

$$\phi P[Y,Z] = A_{\phi Y}Z - A_{\phi Z}Y + \eta(Y)Z - \eta(Z)Y$$

and

$$\eta(Z)\phi Y - \eta(Y)\phi Z = 0$$

for all $Y, Z \in D^{\perp}$. Thus we have

Theorem 4.3. Let M be a CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi Z}Y - A_{\phi Y}Z = \eta(Y)Z - \eta(Z)Y$$

for all $Y, Z \in D^{\perp}$.

Theorem 4.4. Let M be a CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then the distribution D is integrable if and only if

$$h(X,\phi Y) = h(Y,\phi X)$$

for all $Y, Z \in D$.

Proof : For $X, Y \in D$, from (3.10) we have

$$h(X,\phi Y) = \phi Q\nabla_X Y + Ch(X,Y),$$

which gives

$$\phi Q[X,Y] = h(X,\phi Y) - h(Y,\phi X).$$

Thus D is integrable if and only if $h(X,\phi Y) = h(Y,\phi X)$.

Definition : The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.

Proposition 4.5. Let M be a ξ-vertical CR-submanifold of a Lorentzian para-Sasakian manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then the distribution $D^{\perp}$ is parallel with respect to the connection ∇ on M, if and only if $A_N X \in D^{\perp}$ for each $X \in D^{\perp}$ and any $N \in TM^{\perp}$.
Proof: Let $Y, X \in D^{\perp}$. Then using (3.5), (3.6) and (3.7), we have

$$-A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \eta(X)\phi Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$$
$$+g(X,\phi Y)\xi + \phi(\nabla_X Y + h(X,Y)).$$

Taking inner product with $Z \in D$, we get

$$g(A_{\phi Y}X, Z) = g(\nabla_X Y, \phi Z).$$

Therefore, $\nabla_X Y = 0$ if and only if $A_{\phi Y}X \in D^{\perp}$ for all $X \in D^{\perp}$. From which our assertion follows.
Definition : A CR-submanifold M of an LP-Sasakian manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection is said to be totally geodesic if $h(X,Y) = 0$ for $X \in D$ and $Y \in D^{\perp}$.

It follows immediately that a CR-submanifold is mixed totally geodesic if and only if $A_N X \in D$ for each $X \in D$ and $N \in T^{\perp}M$.

Let $X \in D$ and $Y \in \phi D^{\perp}$. For a mixed totally geodesic ξ-vertical CR-submanifold M of an LP-Sasakian manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection then from (3.4), we have

$$(\bar{\nabla}_X \phi)N = 0.$$

Since $\bar{\nabla}_X \phi N = (\bar{\nabla}_X \phi)N + \phi(\bar{\nabla}_X N)$ so that $\bar{\nabla}_X \phi N = \phi(\bar{\nabla}_X N)$. Using (3.6) and (3.7) in above equation, we have

$$\nabla_X(\phi N) = -\phi A_N X + \phi \nabla^{\perp}_X N.$$

As $A_N X \in D$, $\phi A_N X \in D$, so $\phi \nabla^{\perp}_X N = 0$ if and only if $\bar{\nabla}_X \phi N \in D$.
Thus we have the following proposition.
Proposition 4.6. Let M be a mixed totally ξ-vertical CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then the normal section $N \in \phi D^{\perp}$ is $D-$parallel if and only if $\nabla_X(\phi N) \in D$ for all $X \in D$.

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