On $Z_3$-Magic Labeling and Cayley Digraphs

J. Baskar Babujee and L. Shobana

Department of Mathematics
Anna University Chennai, Chennai-600 025, India
baskarbabujee@yahoo.com, shobana_2210@yahoo.com

Abstract

Let $A$ be an abelian group. An $A$-magic of a graph $G = (V, E)$ is a labeling $f : E(G) \to A \setminus \{0\}$ such that the sum of the labels of the edges incident with $u \in V$ is a constant, where 0 is the identity element of the group $A$. In this paper we prove $Z_3$-magic labeling for the class of even cycles, Bistar, ladder, biregular graphs and for a certain class of Cayley digraphs.

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1 Introduction

Let $G$ be a connected or multi graph, with no loops. For any additively abelian group $A$, let $A^* = A - \{0\}$. A function $f : E(G) \to A^*$ is called the labeling of $G$. Any such labeling induces a map $f^* : V(G) \to A$ defined by $f^*(v) = \sum f(u, v)$, where $(u, v) \in E(G)$. If there exists a labeling $f$ whose induced map on $V(G)$ is a constant map, we say that $f$ is an $A$-magic labeling and that $G$ is an $A$-magic graph.

The original concept of an $A$-magic graph is due to Sedlacek [9, 10], who defined it to be a graph with real-valued edge labeling such that distinct edges have distinct nonnegative labels. Though different kinds of labeling were studied and many conjectures were made for different subclasses of graphs the labeling of well known graph namely Cayley graphs has not been investigated for $Z_3$-magic labeling.

The Cayley digraph of a group provides a method of visualizing the group and its properties. The properties such as commutativity and the multiplication table of a group can be recovered from a Cayley digraph. A directed graph or digraph is a finite set of points called vertices and a set of arrows called arcs connecting some of the vertices. The idea of representing a group
in such a manner was originated by Cayley in 1878.

The Cayley graphs and Cayley digraphs are excellent models for interconnection networks [1, 2]. Many well-known interconnection networks are Cayley digraphs. For example, hypercube, butterfly and cube-connected cycle's networks are Cayley graphs [5]. The Cayley digraph for a given finite group $G$ with $S$ as a set of generators of $G$ has two properties. The first property is that each element of $G$ is a vertex of the Cayley digraph. The second property is for $a$ and $b$ in $G$, there is an arc from $a$ to $b$ if and only if $as = b$ for some $s$ in $S$.

It is easy to verify whether a graph is $\mathbb{Z}_2$-magic or not. It seems that finding a similar condition for $G$ to be $\mathbb{Z}_3$-magic is much more difficult. The sum of the labels of the edges incident to a particular vertex is the same for all vertices. In this paper we prove that even cycles, complete graphs, Bistar, ladder, a class of biregular graphs and for a certain class of Cayley digraphs admits $\mathbb{Z}_3$-magic labeling.

**Theorem 1.1.** ([7] (Richard.M. Low and S-M. Lee))
Let $G$ be $\mathbb{Z}_3$-magic, with $p$ vertices and $q$ edges. Let $f \in [G, \mathbb{Z}_3]$ induce the constant label $x$ on the vertices of $G$, and $|E_i|$ denote the number of edges labeled $i$. Then, $px \equiv q + |E_1| \pmod{3}$.

**Proof.** With any $\mathbb{Z}_3$-magic labeling $f$ of $G$, we can associate a multigraph $\hat{G}$ with $G$. $\hat{G}$ is formed by replacing every edge in $G$ which was labeled 2, with two edges labeled 1. Note that $\hat{G}$ is a $\mathbb{Z}_3$-magic multi-graph with $p$ vertices and $2|E_2| + |E_1|$ edges. In any $(p, q)$-graph, we have that $\sum \deg(vi) = 2q$. Since all of the edges in $\hat{G}$ are labeled 1, this implies that $px \equiv |E_2| + |E_1| \pmod{3}$. Thus, $px \equiv q + |E_1| \pmod{3}$. \hfill $\Box$

Let $A$ be an abelian group of even order. A graph $G$ is $A$-magic if degrees of its vertices are either all odd or all even.

**Proof.** Let $a$ be an element of $A$ of order 2. We label all the edges of $G$ by $a$. Then $l^*(v) = a$ or 0 for all $v \in V(G)$ as the degree of $v$ is odd or even respectively. \hfill $\Box$

## 2 $\mathbb{Z}_3$-Magic Labeling

**Example 2.1.** The even cycle graph $C_n$, $n \equiv 0 \pmod{2}$ is $\mathbb{Z}_3$-magic. The Figure 1 illustrates $\mathbb{Z}_3$-magicness of $C_6$. 
Algorithm 2.2.

**Input:** $n$, the number of vertices of cycle graph $C_n$.

**Output:** Construction of $C_n$ along with $Z_3$-magic labeling.

**Begin**

1. $V = \{v_1, v_2, \ldots, v_n\}$
2. $E = \{v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_n v_1\}$
3. For $1 \leq i \leq n - 1$, $f(v_i v_{i+1}) = \begin{cases} 1 & \text{if } i \equiv 1 \mod 2 \\ 2 & \text{if } i \equiv 0 \mod 2 \end{cases}$
   
   and $f(v_n v_1) = 2$.

**End.**

**Theorem 2.3.** The cycle graph $C_n$ admits $Z_3$-magic labeling for $n$ even.

**Proof.** The above Algorithm 2.2 generates a mapping $f : E \rightarrow Z_3 - \{0\}$ to the cycle graph $C_n$. We need to prove that the induced mapping $f^+ : V \rightarrow Z_3$ is constant. For any vertex $v_k \in V$ and $2 \leq k \leq n - 1$,

$$f^+(v_k) = v_k v_{k+1} + v_{k-1} v_k = \begin{cases} 1 + 2 & \text{if } k \text{ is odd} \\ 2 + 1 & \text{if } k \text{ is even} \end{cases} \equiv 0 \mod 3.$$

Also $f^+(v_n) = v_n v_1 + v_{n-1} v_n = 2 + 1 \equiv 0 \mod 3$ and $f^+(v_1) = v_1 v_2 + v_n v_1 = 1 + 2 \equiv 0 \mod 3$.

Hence $f^+(v_k) = 0$ for $\forall k$ and $C_n : n$ even is $Z_3$-magic.

Algorithm 2.4.

**Input:** $n$, the number of vertices of complete graph $K_n$.

**Output:** $Z_3$-magic labeling for $K_n$.

**Begin**

1. $V = \{v_1, v_2, \ldots, v_n\}$
2. $E = \{v_i v_j : 1 \leq i \leq n, 1 \leq j \leq n : i < j\}$
Step 3 : \( f(v_i v_{i+1}) = f(v_i v_{i+1}) = \begin{cases} 1 & 1 \leq i \leq n - 1 \\ 2 & \text{otherwise} \end{cases} \)

End

**Theorem 2.5.** For \( n \geq 4 \), the complete graph \( K_n \) is \( Z_3 \)-magic.

**Proof.** Consider the mapping \( f : E \rightarrow Z_3 - \{0\} \) then we have to prove that the induced mapping \( f^+ : V \rightarrow Z_3 \) is constant. Since complete graphs are pairwise adjacent, we define \( f^+(v_k) \) to be \( f^+(v_k) = \sum_{j=1,j\neq k}^{n} v_k v_j \equiv (2n-4) \mod 3 \).

**Case 1:**
When \( n = 4, 7, 10, \ldots, (3k + 1) \) then the \( Z_3 \)-magic labeling for \( K_4, K_7, K_{10}, \ldots, K_{3k+1} \) is defined as follows. The induced map \( f^+ \) is given by \( f^+ \equiv (2n - 4) \mod 3 \). When \( n = 3k + 1 \), then using Algorithm 2.4.

\[
\begin{align*}
f^+(v_k) &= 2(3k + 1) - 4 \\
&= 6k + 2 - 4 \\
&\equiv 1 \mod 3.
\end{align*}
\]

Hence \( Z_3 \)-magic labeling for \( n = 3k + 1 \) is constantly 1.

**Case 2:**
When \( n = 5, 8, 11, \ldots, (3k + 2) \), the induced mapping \( f^+ \) is given by,

\[
\begin{align*}
f^+(v_k) &= 2(3k + 2) - 4 \\
&= 6k + 4 - 4 \\
&\equiv 0 \mod 3.
\end{align*}
\]

Therefore the \( Z_3 \)-magic labeling for \( n = 3k + 2 \) is 0.

**Case 3:**
When \( n = 6, 9, 12, \ldots, (3k + 3) \), the induced mapping \( f^+ \) is given by,

\[
\begin{align*}
f^+(v_k) &= 2(3k + 3) - 4 \\
&= 6k + 6 - 4 \\
&\equiv 2 \mod 3.
\end{align*}
\]

Therefore \( f^+(v_k) \) is constantly 2 for \( \forall k \).

Hence the complete graph \( K_n \) is \( Z_3 \)-magic.

**Theorem 2.6.** The Bistar \( B_{3k-1,3k-1} \forall k \geq \) is \( Z_3 \)-magic.
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Proof. \( V = \{v_1, v_2, \ldots, v_{2n+2}\}, \)
\( E = \{v_1v_{n+2}\} \cup \{v_iv_j : 2 \leq i \leq n + 1\} \cup \{v_{n+2}v_j : n + 3 \leq j \leq 2n + 2\}, \)
\( f(v_i) = f(v_{n+2}) = 2 \) for \( 2 \leq i \leq n + 1 \) and \( n + 3 \leq j \leq 2n + 2 \) and
\( f(v_{v_{n+2}}) = 1. \)
The induced map \( f^+ \) is given by
\[
\begin{align*}
f^+(v_k) &= 2(3k - 1) + 1 \\
&= 6k - 2 + 1 \\
&\equiv 2 \mod 3.
\end{align*}
\]
Therefore \( f^+(v_k) \) is constantly 2 for \( \forall \ k. \)
Hence the Bistar \( B_{3k-1,3k-1} \forall k \geq 1 \) is \( Z₃ \)-magic.

**Theorem 2.7.** The Ladder \( L_n, n \geq 4 \) is \( Z₃ \)-magic.

Proof. **Case 1:** \( n \) even
Suppose \( n \) is even, \( n \geq 4 \) then the edges are labeled in the following way:
Begin
Step 1: \( V = \{v_1, v_2, \ldots, v_2n\} \)
Step 2: \( E = \{v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_{j+1}v_j : n + 1 \leq j \leq 2n - 1\} \)
∪ \( \{v_kv_{n+k} : 1 \leq k \leq n\} \)
Step 3: \( f(v_i) = f(v_{n+i}) = \begin{cases} 2 & i \equiv 1 \mod 2 \\ 1 & i \equiv 0 \mod 2 \end{cases} \)
Step 4: \( f(v_{n+i}) = \begin{cases} 1 & 2 \leq i \leq n - 1 \\ 2 & i = 1, n \end{cases} \)
End
The induced map \( f^* \) is defined by
\( f^+(v_1) = f^+(v_{n+1}) = f^+(v_n) = f^+(v_{2n}) = 2 + 2 \equiv 1 \mod 3 \) and
\( f^+(v_{2k}) = f^+(v_{2k+1}) = 2 + 1 + 2, \quad k = 1, 2, \ldots, n - 1 \) and \( k \neq \frac{n}{2} \equiv 1 \mod 3 \)
\( f^+(v_{2k+1}) = 1 + 1 + 2, \quad k = 1, 2, \ldots, n - 1 \) and \( k \neq \frac{n}{2} \equiv 1 \mod 3 \)
Therefore when \( n \) is even, then in ladder \( L_n, f^+(v_k) \) is constantly 1 for \( \forall \ k. \)

**Case 2:** \( n \) odd
When \( n \) is odd, the \( Z₃ \)-magic labeling for the ladder \( L_n \) is defined as follows:
Begin
Step 1: \( V = \{v_1, v_2, \ldots, v_{2n}\} \)
Step 2: \( E = \{v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_{j+1}v_j : n + 1 \leq j \leq 2n - 1\} \)
∪ \( \{v_kv_{n+k} : 1 \leq k \leq n\} \)
Step 3: \( f(v_i) = f(v_{n+i}) = 2 \) when \( 2 \leq i \leq n - 2 \) and
\( f(v_{n+1}) = f(v_{n+1}) = f(v_{n+2}) = f(v_{2n-1}) = 1 \)
Step 4: \( f(v_{n+1}) = 1, 3 \leq i \leq n - 2, f(v_{2n+2}) = f(v_{n-1}) = 2 \) and
The induced map \( f^+ \) is given by
\[
f^+(v_1) = f^+(v_n) = f^+(v_{n+1}) = f^+(v_{2n}) = 1 + 1 \equiv 2 \mod 3 \\
f^+(v_i) = 1 + 2 + 2 \equiv 2 \mod 3, i = 2, n + 2 \\
= 2 + 2 + 1 \equiv 2 \mod 3, i = n - 1, 2n - 1 \\
f^+(v_i) = f^+(v_{i+n}) = 2 + 1 + 2 \equiv 2 \mod 3, i = 3, 4, \ldots, n - 2 \\
\]
Therefore \( f^+(v_i) \) is constantly 2 for every \( i \).

Hence the ladder \( L_n, n \geq 4 \) is \( \mathbb{Z}_3 \)-magic.

**Algorithm 2.8.**

**Input:** \( m, n \), the number of vertices in the Biregular bipartite graph.

**Output:** Construction of the \( \mathbb{Z}_3 \)-magic labeling for the Biregular bipartite graph.

**Begin**

**Step 1:** \( V = V_1 \cup V_2 \) where \( V_1 = \{u_1, u_2, \ldots, u_m\} \) and \( V_2 = \{v_1, v_2, \ldots, v_{2n}\} \)

**Step 2:** \( E = E_1 \cup E_2 \cup E_3 \)

\( E_1 = \{v_k u_k, v_k u_{k+1}, k = 1, 2, \ldots, m - 1\} \)

\( E_2 = \{v_m u_m, v_m u_1, v_{2m} u_1\} \) and

\( E_3 = \{v_j u_p, v_j u_{p+1} : 1 \leq p \leq m - 1, m + 1 \leq j \leq 2m - 1\} \)

**Step 3:**

\( f(v_k u_k, v_j u_q) = 1, k = 1, 2, \ldots, m - 1; m + 1 \leq j \leq 2m - 1, 2 \leq q \leq m - 1 \)

\( f(v_k u_{k+1}, v_j u_q) = 2, k = 1, 2, \ldots, m - 1; m + 1 \leq j \leq 2m - 1, 1 \leq q \leq m - 1 \)

\( f(v_m u_1, v_{2m} u_m) = 1 \) and

\( f(v_m u_1, v_{2m} u_1) = 2 \)

**End**

**Figure 2:** The biregular bipartite graph \( K_{3,6} \)

**Theorem 2.9.** The biregular bipartite graph \( K_{m,n} \), \( m \geq 3 \) and \( n = 2m \) is \( \mathbb{Z}_3 \)-magic.

**Proof.** For the \((2,4)\)-biregular bipartite graph \( K_{m,n} \), \( m \geq 3 \) and \( n = 2m \), \( \deg(u_i) = 4 \) and \( \deg(v_j) = 2 \). By the above Algorithm 2.8 we assign labels
for all the edges. We have to prove that the induced map $f^+ : V \to \mathbb{Z}_3$ is constant. The induced map $f^+$ is given by
\[
f^+(v_k) = v_ku_k + v_{k+1} = 1 + 2 \equiv 0 \mod 3, k = 1, 2, \ldots, m - 1 \text{ for all edges } v_ku_i \in E_1.
\]
\[
f^+(v_m) = v_mu_m + v_1u_1 = 2 + 1 \equiv 0 \mod 3 \text{ and }
\]
\[
f^+(v_{2m}) = v_{2m}u_m + v_{2m+1} = 1 + 2 \equiv 0 \mod 3 \text{ for all edges } v_ku_j \in E_2.
\]
\[
f^+(v_k) = v_ku_i + v_{k+1} = 1 + 2 \equiv 0 \mod 3, k = m + 1, m + 2, \ldots, 2m - 1 \text{ and }
\]
\[
i = 1, 2, \ldots, m - 1 \text{ for all edges } v_ku_l \in E_3.
\]

Hence the biregular bipartite graph $K_{m,n}$, $m \geq 3$ and $n = 2m$ is $\mathbb{Z}_3$-magic.

3 $\mathbb{Z}_3$-Magic Labeling for Cayley Digraphs

**Definition 3.1.** Let $G$ be a finite group, and let $S$ be a generating subset of $G$. The Cayley digraph $\text{Cay}(G; S)$ is the digraph whose vertices are the elements of $G$, and there is an edge from $g$ to $gs$ whenever $g \in G$ and $s \in S$. If $S = S^{-1}$, then there is an arc from $g$ to $gs$ if and only if there is an arc from $gs$ to $g$.

We present an Algorithm 3.2 to get $\mathbb{Z}_3$-magic labeling for Cayley digraph $\text{Cay}(D_n, (a, b))$.

**Algorithm 3.2.**

**Input:** The dihedral group $D_n$ with the generating set $(a, b)$.

**Begin**

**Step 1:** Using Definition 3.1, construct the Cayley digraph $\text{Cay}(D_n, (a, b))$.

**Step 2:** Denote the vertex set of $\text{Cay}(D_n, (a, b))$ as $V = \{v_1, v_2, v_3, \ldots, v_n\}$.

**Step 3:** Denote the edge set of $\text{Cay}(D_n, (a, b))$ as $E = E_a \cup E_b$ where
\[
E_a = \text{Set of all going arcs from } v_i \text{ generated by } a
\]
\[
E_b = \text{Set of all going arcs from } v_i \text{ generated by } b
\]

**Step 4:** Define $f$ such that $f(v_i) = i$, for $1 \leq i \leq n$.

**Step 5:** Define $g_a$ such that $g_a(v_i) = \begin{cases} 2; & 1 \leq i \leq n/2 \\ 1; & (n/2) + 1 \leq i \leq n \end{cases}$

**Step 6:** Define $g_b$ such that $g_b(v_i) = \begin{cases} 2; & 1 \leq i \leq n/2 \\ 1; & (n/2) + 1 \leq i \leq n \end{cases}$

**End**

**Output:** $\mathbb{Z}_3$-magic labeling for Cayley digraph $\text{Cay}(D_n, (a, b))$.

**Theorem 3.3.** The Cayley digraph associated with dihedral group $D_n$ admits $\mathbb{Z}_3$-magic labeling.

**Proof.** From the construction of the Cayley digraph for the dihedral group $D_n$, we have $\text{Cay}(D_n, (a, b))$ has $n$ vertices and $2n$ arcs. Let us denote the vertex set
of $\text{Cay}(D_n, (a, b))$ as $V = \{v_1, v_2, v_3, \ldots, v_n\}$. To prove $\text{Cay}(D_n, (a, b))$ admits $Z_3$-magic labeling we have to show that for any vertex $v_i$, the sum of the labels of the outgoing arcs are constant. By construction of the Cayley digraph, we have each vertex has exactly two outgoing arcs out of which one is from the set $E_a$ and another is from the set $E_b$, where $E_a$ and $E_b$ are defined in the above algorithm. Now define maps $f$, $g_a$ and $g_b$ as defined in steps 4, 5 and 6 of the above algorithm. Hence $g_a(v_i) + g_b(v_i) = 1 + 2 = 2 + 1 = 0 \mod 3$. Therefore the Cayley digraph $\text{Cay}(D_n, (a, b))$ is $Z_3$-magic. 

**Example 3.4.** Consider the group dihedral group $D_{10} = \{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$ with the generating set $A = \{a, b\}$ such that $a^5 = b^2 = 1$. The Cayley digraph for the dihedral group $D_{10}$ and its $Z_3$-magic labeling is shown in Figure 3.

![Figure 3: Z_3-magic labeling of the Cayley digraph for the dihedral group D_{10}](image)

**Definition 3.5.** For any natural number $n$, we use $Z_n$ to denote the additive cyclic group of integers modulo $n$. In this section we consider the group $(Z_{2k}, A)$ where $A$ is a generating set $\{a, b, b+k\}$ with the property that

- $\gcd(a, b, k) = 1$ and either
- $\gcd(a-b, k) \neq 1$ or $\gcd(a, 2k) = 1$ or
- $\gcd(b, k) = 1$ or
- both $a$ and $k$ are even or
- $a$ is odd and either $b$ or $k$ is odd.

**Algorithm 3.6.**

**Input:** The group $Z_n$ with the generating set $(a, b, b+k)$.

**Begin**
Step 1: Construct the Cayley digraph $\text{Cay}(Z_n, (a, b, b + k))$, using Definition 3.5.

Step 2: Denote the vertex set of $\text{Cay}(Z_n, (a, b, b+k))$ as $V = \{v_1, v_2, v_3, \ldots, v_n\}$

Step 3: Denote the edge set of $\text{Cay}(Z_n, (a, b, b + k))$ as $E(E_a, E_b, E_c) = \{e_1, e_2, e_3, \ldots, e_{3n}\}$ where

$E_a = \text{Set of all out going arcs from } v_i \text{ generated by } a$

$E_b = \text{Set of all out going arcs from } v_i \text{ generated by } b$

$E_c = \text{Set of all out going arcs from } v_i \text{ generated by } b + k$

Step 4: Define $f$ such that $f(v_i) = i + 1$, for $0 \leq i \leq n - 1$

Step 5: Define $g_a$ such that $g_a(v_i) = 2; 0 \leq i \leq n - 1$

Step 6: Define $g_b$ such that $g_b(v_i) = 2; 0 \leq i \leq n - 1$

Step 7: Define $g_c$ such that $g_c(v_i) = 1; 0 \leq i \leq n - 1$

End

Output: $Z_3$-magic labeling for Cayley digraph $\text{Cay}(Z_n, (a, b, b + k))$.

**Theorem 3.7.** The Cayley digraph associated with the group $Z_n$ with the generating set $(a, b, b + k)$ admits $Z_3$-magic labeling.

**Proof.** From the construction of the cayley digraph for the group $Z_n$, we have $\text{Cay}(Z_n, (a, b, b + k))$ has $n$ vertices and $3n$ arcs. Let us denote the vertex set of $\text{Cay}(Z_n, (a, b, b + k))$ as $V = \{v_1, v_2, v_3, \ldots, v_n\}$ corresponding to the elements $\{0, 1, 2, 3, \ldots, n - 1\}$ of the group $Z_n$ respectively and the edge set of $\text{Cay}(Z_n, (a, b, b + k))$ as $E(E_a, E_b, E_c) = \{e_1, e_2, e_3, \ldots, e_{3n}\}$.

To prove $\text{Cay}(Z_n, (a, b, b + k))$ admits $Z_3$-magic labeling we have to show that for any vertex $v_i$, the sum of the labels of the outgoing arcs are constant. Define the edges of $\text{Cay}(Z_n, (a, b, b + k))$ as follows:

For each $i \in Z_n$, $j \in A$ define a directed edge $v_i$ to $v_{i+j}(mod n)$. Consider an arbitrary vertex $v_i \in V$ of the cayley digraph $\text{Cay}(Z_n, (a, b, b + k))$. By construction of the cayley digraph we have each vertex has exactly three outgoing arcs which are from the set $E_a, E_b, E_c$ respectively where

$E_a = \{(v_i, v_{(a+i)}(mod n))|0 \leq i \leq n - 1\}$

$E_b = \{(v_i, v_{(b+i)}(mod n))|0 \leq i \leq n - 1\}$

$E_c = \{(v_i, v_{(b+k+i)}(mod n))|0 \leq i \leq n - 1\}$

Now define maps $f, g_a, g_b, g_c$ as defined in steps 4, 5, 6 and 7 of the above algorithm. Thus $g_a(v_i) + g_b(v_i) + g_c(v_i) = 2 + 2 + 1 = 2 mod 3$. Hence the cayley digraph associated with dihedral group $Z_n$ admits $Z_3$-magic labeling. \(\square\)

**Example 3.8.** Consider the group $(Z_5; 5; 3; 7)$ consisting of 8 elements where $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and the generating set $A = \{5, 3, 7\}$. The cayley digraph for the group $(Z_5; 5, 3, 7)$ and its $Z_3$-magic labeling is shown in Figure 4.

**Example 3.9.** Consider the group $S_3$, consisting of 6 elements. The Cayley digraph for the symmetric group $S_3$ is shown in Figure 5.
Algorithm 3.10.

**Input:** The symmetric group $S_k$ with the generating set $(\tau, \sigma)$.

**Begin**

Step 1: Using Definition 3.1, construct the Cayley digraph $\text{Cay}(S_k, (\tau, \sigma))$.

Step 2: Denote the vertex set of $\text{Cay}(S_k, (\tau, \sigma))$ as $V = \{v_1, v_2, v_3, \ldots, v_n\}$ where $n = k!$

Step 3: Denote the edge set of $\text{Cay}(S_k, (\tau, \sigma))$ as $E = E_\tau, E_\sigma = \{e_1, e_2, e_3, \ldots, e_{2n}\}$ where $E_\tau$ = Set of all outgoing arcs from $v_i$ generated by $\tau$ and $E_\sigma$ = Set of all outgoing arcs from $v_i$ generated by $\sigma$

Step 4: Define $f$ such that $f(v_i) = i$, for $1 \leq i \leq n$

Step 5: Define $g_\tau$ such that $g_\tau(v_i) = 2$, for $1 \leq i \leq n$

Step 6: Define $g_\sigma$ such that $g_\sigma(v_i) = 1$, for $1 \leq i \leq n$

**End**

**Output:** $Z_3$-magic labeling for the Cayley digraph $\text{Cay}(S_k, (\tau, \sigma))$.

**Theorem 3.11.** The Cayley digraph associated with symmetric group $S_k$ ad-

Figure 4: $Z_3$-magic labeling of the Cayley digraph for the group $(Z_8; 5, 3, 7)$

Figure 5: $Z_3$-magic labeling of the Cayley digraph for symmetric group $S_3$
mits $Z_3$-magic labeling.

Proof. From the construction of the Cayley digraph for the symmetric group $S_k$, we have $Cay(S_k, (\tau, \sigma))$ has $n = k!$ vertices and $2n$ arcs. Let us denote the vertex set of $Cay(S_k, (\tau, \sigma))$ as $V = \{v_1, v_2, v_3, \ldots, v_n\}$. To prove $Cay(S_k, (\tau, \sigma))$ admits $Z_3$-magic labeling we have to show that for any vertex $v_i$, the sum of the label of its outgoing arcs are constant. Consider an arbitrary vertex $v_i \in V$ of the Cayley digraph $Cay(S_k, (\tau, \sigma))$. By construction of the Cayley digraph we have each vertex has exactly two outgoing arcs out of which one arc is from the set $E_\tau$ and another is from the set $E_\sigma$, where $E_\tau$ and $E_\sigma$ are as defined in the above algorithm. Now define maps $f, g_\tau$ and $g_\sigma$ as defined in steps 4, 5 and 6 of the above algorithm. Hence $g_\tau(v_i) + g_\sigma(v_i) = 2 + 1 \equiv 0 \mod 3$.

Hence the Cayley digraph associated with symmetric group $S_k$ admits $Z_3$-magic labeling. \qed

References


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