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Approximation Solution of Second Order Initial Value Problem by Spline Function of Degree Seven

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Abstract

In the present paper we found approximation solution of second order initial value problem by seven degree lacunary spline function of the type (0, 1, 4) .Some illustrative examples are presented with using SSF algorithm for calculating absolute error between spline functions and exact solution of second order initial value problem and also their derivatives .

Keywords: Seven degree spline, second order Initial value problem, approximation solution

1. INTRODUCTION

The literature on the numerical solutions of initial value problems by using lacunary spline functions is not too much. Gyovari [3] solved Cauchy problem by sing modified lacunary spline function which interpolating the lacunary data (0, 2, 3).

Saxena[7], also used deficient lacunary spline for solving Cauchy problem too.Saxena and Venturino [8], treated two-points boundary value problem by using lacunary spline function which interpolates the lacunary data (0, 2). Eamail, *et al*[2] used lacunary spline function to solve fourth order differential equations.Abbas, *et al* [1] and Jwamer [4] found the approximations solution for different lacunary interpolations which are (0,3,5) and (0,3,4) by spline functions of degree six . Siddiqi, *et al*[9] used quintic spline for solving six-order boundary value problem.

In this work, we try to solve the initial value problem

$$y'' = f(x, y, y') \ , y(x_0) = y_1 \ , y'(x_0) = y'_2 \qquad (1.1)$$

using that $f \in C^{n-1}([0,1]\times R^2)$, $n \geq 2$ and that it satisfies the Lipschitz condition

$$\left| f^{(q)}(x, y_1, y_1') - f(x, y_2, y_2') \right| \leq L\{|y_1 - y_2| + |y_1' - y_2'|\}, \ q=0,1,\ldots, n-1$$

for all $x \in [0,1]$ and all reals y_1, y_2, y_1', y_2'. These conditions ensure the existence of the unique solution of the problem (1.1).

The lacunary interpolation problem, which we have investigated in this work, consists in finding the seven degree spline S(x) of deficiency four, interpolating data given on the function value and one and fourth derivatives in the interval [0, 1]. Also, an extra initial condition is prescribed on the two and three derivatives.

This paper is organized as follows:

In section 2, spline function of degree seven is presented which interpolates the lacunary data (0, 1, 4). Some theoretical results about existence and uniqueness of the spline function of degree seven are introduced in section 3. In section 4 , convergence and error bound are studied. To demonstrate the convergence of the prescribed lacunary spline function, numerical examples presented in section 5.

2. DESCRIPTIONS OF THE METHOD

In this section , we present a seven degree spline interpolation for one dimensional and given sufficiently smooth function $f(x)$ defined on I=[0,1], and $\Delta_n: \ 0 = x_0 < x_1 < x_2 < \ldots . < x_n = 1$, denote the uniform partition of I with knots $x_i = ih$, ,where $i = 0,1,2,\ldots,n-1$ and h is the distance of each subintervals. It is denoted by $S^4_{n,7}$ the class of seven degree splines $S(x)$ as:

$$s_0(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2}y_0'' + \frac{(x-x_0)^3}{6}y_0''' + \frac{(x-x_0)^4}{24}y_0^{(4)} + (x-x_0)^5 a_{0,5} + (x-x_0)^6 a_{0,6} + (x-x_0)^7 a_{0,7} \qquad (2.1)$$

on the subintervals $[x_0, x_1]$ where $a_{0,j}$, $j = 5,6,7$ are unknowns to be determined.

Let us examine subintervals $[x_i, x_{i+1}]$, i=1,2,…, n-2. By taking into account the interpolating conditions, we can write the expression, for $S_i(x)$ in the following form, see Saeed [5] and Saeed and Jwamer [6] .

$$s_i(x) = y_i + (x-x_0)y_i' + (x-x_i)^2 a_{i,2} + (x-x_i)^3 a_{i,3} + \frac{(x-x_i)^4}{24}y_i^{(4)} + (x-x_i)^5 a_{i,5} + (x-x_i)^6 a_{i,6} + (x-x_i)^7 a_{i,7} \qquad (2.2)$$

where $a_{i,j}$, $i = 1(1)(n-1)$, $j = 2,3,5,6,7$ are unknown values we need to determine it.

3. THEORETICAL RESULTS

In this section, the existence and uniqueness theorem for spline function of degree seven which interpolate the lacunary data (0, 1, 4) is presented and examined.

Theorem 3.1: New Existence and Uniqueness of the Spline Function

Given the real numbers $y(x_i)$, $y'(x_i)$ and $y^{(4)}(x_i)$ for i=0, 1, 2, …, n, then there exist a unique spline of degree seven as given in the equations (2.1)-(2.2) such that:

$$\left.\begin{aligned} &S(x_i)=y(x_i)\\ &S^{(r)}(x_i)=y^{(r)}(x_i),\ r=1,4\\ &\text{and}\\ &S''(x_0)=y''(x_0)\ \textit{and}\ S'''(x_0)=y'''(x_0)\end{aligned}\right\}\text{for } i=0,1,\dots,n \tag{3.1}$$

Proof:

The spline function S(x) is defined as follows:

$$S(x)=\begin{cases} S_0(x) & \text{when } x\in[x_0,x_1]\\ & \qquad ;\ i=0,1,\dots,n-2\\ S_i(x) & \text{when } x\in[x_i,x_{i+1}]\end{cases}$$

where the coefficients of these polynomials are to be determined by the following conditions:

$$\left.\begin{aligned} &S_i(x_{i+1})=S_{i+1}(x_{i+1})=y_{i+1}\\ &S_i^{(r)}(x_{i+1})=S^{(r)}{}_{i+1}(x_{i+1})=y_{i+1}^{(r)},\ r=1,4\\ &S_i''(x_{i+1})=S_{i+1}''(x_{i+1})\ \textit{and}\ S_i'''(x_{i+1})=S_{i+1}'''(x_{i+1})\end{aligned}\right\};\ i=0,1,\dots,n-2, \tag{3.2}$$

$$S_{n-1}(x_n)=y_n,\ S_{n-1}{}^{(r)}(x_n)=y_n{}^{(r)};\ r=1,4 \tag{3.3}$$

To find uniquely the coefficients in $S_0(x)$ of equation (2.1) by using the condition (3.2) where i=0, we obtain the following:

$$h^5a_{0.5}+h^6a_{0.6}+h^7a_{0.7}=y_1-y_0-hy_0'-\frac{h^2}{2}y_0''-\frac{h^3}{6}y_0^{(3)}-\frac{h^4}{24}y_0^{(4)},$$

$$5h^4a_{0.5}+6h^5a_{0,6}+7h^6a_{0,7}=y_1'-y_0'-hy_0''-\frac{h^2}{2}y_0'''-\frac{h^3}{6}y_0^{(4)},$$

And $120ha_{0.5}+360h^2a_{0,6}+840h^3a_{0,7}=y_1^{(4)}-y_0^{(4)}$.

From the boundary condition (3.3) we have

$$20h^3a_{0.5}+30h^4a_{0.6}+42h^5a_{0.7}=2a_{1,2}-y_0''-hy_0^{(3)}-\frac{h^2}{2}y_0^{(4)}. \tag{3.4}$$

$$60h^2a_{0.5}+120h^3a_{0.6}+210h^4a_{0.7}=6a_{1,3}-y_0^{(3)}-hy_0^{(4)} \tag{3.5}$$

Solving these equations to obtain the following:

$$a_{0,5}=\frac{21}{2h^5}[y_1-y_0]-\frac{1}{2h^4}[4y_1'-17y_0']+\frac{1}{240h}[y_1^{(4)}-26y_0^{(4)}]$$
$$-\frac{13}{4h^3}y_0''-\frac{3}{4h^2}y_0^{(3)} \tag{3.6}$$

$$a_{0,6}=\frac{-14}{h^6}[y_1-y_0]+\frac{1}{h^5}[3y_1'+11y_0']-\frac{1}{120h^2}[y_1^{(4)}-11y_0^{(4)}]$$
$$+\frac{4}{h^4}y_0''+\frac{5}{6h^3}y_0^{(3)}, \tag{3.7}$$

and also, we get

$$a_{0,7} = \frac{9}{2h^7}[y_1 - y_0] - \frac{1}{2h^6}[2y_1' + 7y_0'] + \frac{1}{240\,h^3}[y_1^{(4)} - 6y_0^{(4)}]$$
$$-\frac{5}{4h^5}y_0'' - \frac{1}{4h^4}y_0^{(3)} \qquad (3.8)$$

Substituting these values of $a_{0,5}$, $a_{0,6}$ and $a_{0,7}$ in equation (3.4) and (3.5) we get:

$$a_{1,2} = \frac{-21}{2h^2}[y_1 - y_0] + \frac{1}{2h}[8y_1' + 13y_0'] + \frac{h^2}{240}[y_1^{(4)} + 4y_0^{(4)}]$$
$$+\frac{7}{4}y_0'' + \frac{h}{4}y_0^{(3)} \qquad (3.9)$$

and also, we get

$$a_{1,3} = \frac{-35}{2h^3}[y_1 - y_0] + \frac{1}{2h}[10\ y_1' + 25\ y_0'] + \frac{h}{48}[y_1^{(4)} + 2y_0^{(4)}]$$
$$+\frac{15}{4}y_0'' + \frac{7h}{12}y_0^{(3)} \qquad (3.10)$$

We shall find the coefficients of $S_i(x)$ for $i = 1, 2, 3, ..., n-1$. From equation (2.2) we have:

$$h^2 a_{i,2} + h^3 a_{i.3} + h^5 a_{i.5} + h^6 a_{i.6} + h^7 a_{i,7} = y_{i+1} - y_i - hy_i' - \frac{h^4}{24}y_i^{(4)},$$

$$2ha_{i,2} + 3h^2 a_{i.3} + 5h^4 a_{i.5} + 6h^5 a_{i.6} + 7h^6 a_{i,7} = y_{i+1}' - y_i' - \frac{h^3}{6}y_i^{(4)},$$

$$120\,ha_{i.5} + 360\,h^2 a_{i.6} + 840\,h^3 a_{i,7} = y_{i+1}^{(4)} - y_i^{(4)},$$

and form $s_k''(x_{k+1}) = y_{k+1}''$ and $s_k'''(x_{k+1}) = y_{k+1}'''$, we have

$$s_i''(x_{ik+1}) = y_{i+1}'' = 2a_{i+1,2} = 2a_{i,2} + 6ha_{i,3} + \frac{h^2}{2}y_i^{(4)} + 20h^3 a_{i,5} + 30h^4 a_{i,6} + 42h^5 a_{i,7}$$

$$a_{i,2} - a_{i+1,2} + 3ha_{i,3} + 10h^3 a_{i,5} + 15h^4 a_{i,6} + 21h^5 a_{i,7} = -\frac{h^2}{4}y_i^{(4)} \qquad (3.11)$$

$$s_i'''(x_{ik+1}) = y_{i+1}''' = 6a_{i+1,3} = 6a_{i,3} + 60h^2 a_{i,5} + 120h^3 a_{i,6} + 120h^4 a_{i,7} + hy_i^{(4)}$$

$$a_{i,3} - a_{i+1,3} + 10h^2 a_{i,5} + 20h^3 a_{i,6} + 35h^4 a_{i,7} = -\frac{h}{6}y_i^{(4)} \qquad (3.12)$$

Solving the first three equations, we obtain the following:

$$a_{i,5} = \frac{21}{2h^5}[y_{i+1} - y_i] - \frac{1}{2h^4}[4y_{i+1}' + 17\ y_i'] + \frac{1}{240\ h}[y_{i+1}^{(4)} + 26\ y_i^{(4)}]$$
$$-\frac{13}{2h^3}a_{i,2} - \frac{9}{2h^2}a_{i,3} \qquad (3.13)$$

$$a_{i,6} = \frac{-14}{h^6}[y_{i+1} - y_i] + \frac{1}{h^5}[3y_{i+1}' + 11\ y_i'] - \frac{1}{120\ h^2}[y_{i+1}^{(4)} + 11\ y_i^{(4)}]$$
$$+\frac{8}{h^4}a_{i,2} + \frac{5}{h^3}a_{i,3} \qquad (3.14)$$

and

$$a_{i,7}=\frac{9}{2h^7}[y_{i+1}-y_i]-\frac{1}{2h^6}[2y'_{i+1}+7y'_i]+\frac{1}{240h^3}[y^{(4)}_{i+1}-6y^{(4)}_i]$$
$$-\frac{5}{2h^5}a_{i,2}-\frac{3}{2h^4}a_{i,3} \qquad (3.15)$$

Substituting the values of $a_{i,2}$, $a_{i,4}$ and $a_{i,6}$ in the fourth equation, we obtain the following relation, for $i=1,2,...,n-1$.

$$2a_{i,2}-2a_{i+1,2}=-6ha_{i,3}-20h^3\left[-\frac{13}{2h^3}a_{i,2}-\frac{9}{2h^2}a_{i,3}+\frac{21}{2h^5}[y_{i+1}-y_i]\right.$$
$$\left.-\frac{1}{2h^4}[4y'_{i+1}+17y'_i]+\frac{1}{240h}[y^{(4)}_{i+1}+26y^{(4)}_i]\right]-30h^4\left[\frac{8}{h^4}a_{i,2}+\right.$$
$$\frac{5}{h^3}a_{i,3}-\frac{14}{h^6}[y_{i+1}-y_i]+\frac{1}{h^5}[3y'_{i+1}+11y'_i]-\frac{1}{120h^2}[y^{(4)}_{i+1}+$$
$$\left.11y^{(4)}_i]\right]-42h^5\left[-\frac{5}{2h^5}a_{i,2}-\frac{3}{2h^4}a_{i,3}\ \frac{9}{2h^7}[y_{i+1}-y_i]-\frac{1}{2h^6}\right.$$
$$\left.[2y'_{i+1}+7y'_i]+\frac{1}{240h^3}[y^{(4)}_{i+1}-6y^{(4)}_i]\right]-\frac{h^2}{2}y^{(4)}_i$$

$$2a_{i+1,2}-7a_{i,2}-3ha_{i,3}=\frac{-21}{h^2}[y_{i+1}-y_i]+\frac{1}{h}[8y'_{i+1}+13y'_i]+\frac{h^2}{120}[y^{(4)}_{i+1}+4y^{(4)}_i]. \qquad (3.16)$$

$$6a_{i,3}-6a_{i+1,3}=-60h^2\left[-\frac{13}{2h^3}a_{i,2}-\frac{9}{2h^2}a_{i,3}+\frac{21}{2h^5}[y_{i+1}-y_i]-\frac{1}{2h^4}\right.$$
$$\left.-\frac{1}{2h^4}[4y'_{i+1}+17y'_i]+\frac{1}{240h}[y^{(4)}_{i+1}+26y^{(4)}_i]\right]-120h^3\left[\frac{8}{h^4}a_{i,2}\right.$$
$$\frac{5}{h^3}a_{i,3}-\frac{14}{h^6}[y_{i+1}-y_i]+\frac{1}{h^5}[3y'_{i+1}+11y'_i]-\frac{1}{120h^2}[y^{(4)}_{i+1}+$$
$$\left.11y^{(4)}_i]\right]-210h^4\left[-\frac{5}{2h^5}a_{i,2}-\frac{3}{2h^4}a_{i,3}\ \frac{9}{2h^7}[y_{i+1}-y_i]-\frac{1}{2h^6}\right.$$
$$\left.[2y'_{i+1}+7y'_i]+\frac{1}{240h^3}[y^{(4)}_{i+1}-6y^{(4)}_i]\right]-hy^{(4)}_i$$

$$6a_{i+1,3}-21a_{i,3}-\frac{45}{h}a_{i,2}=\frac{-105}{h^3}[y_{i+1}-y_i]+\frac{15}{h^2}[2y'_{i+1}+5y'_i]$$
$$+\frac{h^2}{120}[y^{(4)}_{i+1}+4y^{(4)}_i] \qquad (3.17)$$

The coefficient matrix of the system of equations (3.9),(3.10),(3.16)

and (3.17) in the unknown $a_{i,2}$, i=1, 2,…,n-1 is a non-singular matrix and hence the coefficients $a_{i,2}$, i=1, 2,…,n-1 are determined uniquely, and so are , therefore the coefficients $a_{i,3}$, $a_{i,4}$, $a_{i,5}$, $a_{i,6}$ and $a_{i,7}$.
Hence the proof of Theorem 3.1 is completed.

4. CONVERGENCE AND ERROR BOUNDS

In this section, the error bound of the spline function $S(x)$ which is a solution of the problem (3.1) is obtained for the uniform partition of I by the following theorem:

Theorem 4.1

Let $y \in C^7[0,1]$ and $S(x)$ be a unique spline function of degree seven which a solution of the problem (3.1). Then for $x \in [x_0, x_1]$, the following error bounds are holds:

$$\left\| S_0^{(r)}(x) - y^{(r)}(x) \right\| \le \begin{cases} 8w_7(f;h), & r = 0 \\ 11\,hw_7(f;h), & r = 1 \\ \frac{4}{3}h^r w_7(f;h), & r = 2,3 \\ \frac{1}{16}h^4 w_7(f;h), & r = 4 \\ \frac{1}{80}h^r w_7(f;h), & r = 5,6,7 \end{cases}$$

where $W_7(f;h)$ denotes the modules of continuity of $y^{(7)}$, where $\|f(x)\| = \max\{|f(x)|; 0 \le x \le 1\}$.

Proof:

Let $x \in [x_0, x_1]$

We have from equation (2.1) and using Taylor's expansion formula we get

$$S_{0_i}^{(7)}(x) = 5040a_{0,7} \tag{4.1}$$

Using (4.1) and (3.8), we obtain

$$\left| S_0^{(7)}(x) - y^{(7)}(x) \right| = \left| 5040a_{0,7} - y^{(7)}(x) \right| \le 8w_7(f;h) \tag{4.2}$$

From equations (3.7) and (3.8), we obtain

$$S_0^{(6)}(x) = 720\,a_{0,6} + 5040\,h\,a_{0,7}\,,$$

from which we obtain

$$S_0^{(6)}(x) - y^{(6)}(x) = 720a_{0,6} + 5040ha_{0,7} - y^{(6)}(x) \tag{4.3}$$

From equations (4.3), (3.7)and (3.8) ,using Taylor's series expansion on $y^{(6)}(x)$ about $x = x_1$, we get

$$y^{(6)}(x) = y^{(6)}(x_0) + (x - x_0)y^{(7)}(x_0)$$

$$\left|S_0^{(6)}(x)-y^{(6)}(x)\right|=\left|720a_{0,6}+5040ha_{0,7}-y^{(6)}(x_0)+(x-x_0)y^{(7)}(\alpha_1)\right|$$
$$\leq h\left|5040a_{0,7}-y^{(7)}(\alpha_1)\right|+\left|720a_{0,6}-y^{(6)}(x_0)\right| \quad (4.4)$$

where $x_0 < \alpha_1 < x_1$

From (3.7) and using Taylor series expansion, we get

$$\left|720a_{0,6}-y^{(6)}(x_0)\right|=h\left|3y_0^{(7)}-3y_0^{(7)}\right|\leq 3hw_7(f;h) \quad (4.5)$$

From equations (4.2), (4.4)and (4.5) we obtain

$$\left|S_0^{(6)}(x)-y^{(6)}(x)\right|\leq 11hw_7(f;h)$$

To find $\left|S_0^{(5)}(x)-y^{(5)}(x)\right|$, we need the following:

Using Taylor's series expansion on $y^{(5)}(x)$ about $x=x_1$, we get:

$$y^{(5)}(x)=y^{(5)}(x_0)+(x-x_0)y^{(6)}(x_0)+\frac{(x-x_0)^5}{2}y^{(7)}(x_0) \quad (4.6)$$

From equations (2.1), we get:

$$S_0^{(5)}(x)=120\,a_{0,5}+720\,ha_{0,6}+2520\,h^2\,a_{0,7} \quad (4.7)$$

From equation (3.6) - (3.8), (4.6) and (4.7)

$$\left|S_0^{(5)}(x)-y^{(5)}(x)\right|=\left|120a_{0,5}+720ha_{0,6}+2520h^2a_{0,7}-y^{(5)}(x)\right|$$
$$\leq\frac{4}{3}h^2\left|y^{(7)}(\beta_1)-y^{(7)}(\beta_2)\right|\leq\frac{4}{3}h^2w_7(f;h) \quad (4.8)$$

where $x_0<\beta_1,\beta_2<x_1$

By (3.1), $S_0^{(4)}(x_0)-y_0^{(4)}(x_0)=0$, from which we obtain

$$\left|S_0^{(4)}(x)-y^{(4)}(x)\right|=\left|\int_{x_0}^{x}\left(S_0^{(5)}(t)-y^{(5)}(t)\right)dt\right|\leq\int_{x_0}^{x}\left|S_0^{(5)}(t)-y^{(5)}(t)\right|dt$$
$$\leq\int_{x_0}^{x}\frac{4h^2}{3}w_7(f;h)dt=\frac{4h^3}{3}w_7(f;h)$$

To find $\left|S_0^{(3)}(x)-y^{(3)}(x)\right|$, we need the following:

Using Taylor's series expansion on $y^{(3)}(x)$ about $x=x_1$, we get:

$$y^{(3)}(x)=y^{(3)}(x_0)+(x-x_0)y^{(4)}(x_0)+\frac{(x-x_0)^2}{2}y^{(5)}(x_0)+\frac{(x-x_0)^3}{6}y^{(6)}(x_0)+\frac{(x-x_0)^4}{24}y^{(7)}(x_0) \quad (4.9)$$

From equations (3.6)-(3.8), (2.1) and (4.9), we get:

$$\left|S_0^{(3)}(x)-y^{(3)}(x)\right|=\left|y_0^{(3)}+hy_0^{(4)}+60h^2a_{0,5}+120h^3a_{0,6}+210h^4a_{0,7}-y^{(3)}(x_0)-\right.$$
\

$$(x-x_0)y^{(4)}(x_0)-\frac{(x-x_0)^2}{2}y^{(5)}(x_0)-\frac{(x-x_0)^3}{6}y^{(6)}(x_0)-\frac{(x-x_0)^4}{24}y^{(7)}(\delta_2)\Bigg|$$

$$\le \frac{h^4}{16}\left|y^{(7)}(\delta_1)-y^{(7)}(\delta_2)\right| \le \frac{1}{16}h^4 w_7(f;h) \qquad (4.10)$$

where $x_0 < \delta_1, \delta_2 < x_1$.

Using Taylor's series expansion on $y^{(2)}(x)$ about $x = x_1$, we get:

$$y^{(2)}(x) = y^{(2)}(x_0) + (x-x_0)y^{(3)}(x_0) + \frac{(x-x_0)^2}{2}y^{(4)}(x_0) + \frac{(x-x_0)^3}{6}y^{(5)}(x_0)$$

$$+\frac{(x-x_0)^4}{24}y^{(6)}(x_0) + \frac{(x-x_0)^5}{120}y^{(7)}(x_0) \qquad (4.11)$$

From equations (3.6)-(3.8), (2.1) and (4.11), we get:

$$\left|S_0^{(2)}(x)-y^{(2)}(x)\right| = \Bigg| y_0^{(2)} + hy_0^{(3)} + \frac{h^2}{2}y_0^{(4)} + 20h^3 a_{0,5} + 30h^4 a_{0,6} + 42h^5 a_{0,7} - y^{(2)}(x_0)$$

$$-(x-x_0)y^{(3)}(x_0) - \frac{(x-x_0)^2}{2}y^{(4)}(x_0) - \frac{(x-x_0)^3}{6}y^{(5)}(x_0) - \frac{(x-x_0)^4}{24}y^{(6)}(x_0)$$

$$-\frac{(x-x_0)^5}{120}y^{(7)}(\lambda_2)\Bigg| \le \frac{h^5}{80}\left|y^{(7)}(\lambda_1)-y^{(7)}(\lambda_2)\right| \le \frac{1}{80}h^5 w_7(f;h) \qquad (4.12)$$

where $x_0 < \lambda_1, \lambda_2 < x_1$

From (3.1) and (4.12), we have $S_0'(x_0) - y'(x_0) = 0$, from which we obtain

$$\left|S_0'(x) - y_0'(x)\right| = \left|\int_{x_0}^{x}\left(S_0''(t) - y_0''(t)\right)dt\right| \le \int_{x_0}^{x}\left|S_0''(t) - y_0''(t)\right|dt$$

$$\le \frac{1}{80}\int_{x_0}^{x} h^5 w_7(f;h)dt = \frac{h^6}{80}w_7(f;h) \qquad (4.13)$$

Also From (3.1) and (4.13), we have $S_0(x_0) - y(x_0) = 0$, from which we obtain

$$\left|S_0(x) - y_0(x)\right| = \left|\int_{x_0}^{x}\left(S_0'(t) - y_0'(t)\right)dt\right| \le \int_{x_0}^{x}\left|S_0'(t) - y_0'(t)\right|dt$$

$$\le \frac{1}{80}\int_{x_0}^{x} h^6 w_7(f;h)dt = \frac{h^7}{80}w_7(f;h) \qquad (4.14)$$

Theorem 4.2

Let $y \in C^7[0,1]$ and $S(x)$ be a unique spline function of degree seven which a solution of the problem (3.1). Then for $x \in [x_i, x_{i+1}]$; i=1, 2,…, n-1, the following error bounds are holds:

$$\left\| S_i^{(r)}(x) - y^{(r)}(x) \right\| \le \begin{cases} 6300\ K_i w_7(f;h) + 1260\ K_i' w_7(f;h) + 8 w_7(f;h), & r = 0, \\ 9180\ hK_i w_7(f;h) + 1860\ hK_i' h w_7(f;h) + 11\, h\, w_7(f;h), & r = 1, \\ 660\ h^r K_i w_7(f;h) + 120\ h^r K_i' w_7(f;h) + \frac{4}{3} h^r w_7(f;h), & r = 2,3, \\ \frac{45}{2} h^4 K_i w_7(f;h) + \frac{7}{2} h^4 K_i' w_7(f;h) + \frac{1}{16} h^3 w_7(f;h), & r = 4, \\ \frac{7}{2} h^r K_i w_7(f;h) + \frac{1}{2} h^r K_i' w_7(f;h) + \frac{1}{80} h^r w_7(f;h), & r = 5,6,7. \end{cases}$$

To prove this theorem we need Lemma 4.1 and Lemma 4.2 :

Lemma 4.1:

Let $y \in C^7[0,1]$. Then $\left| e_{i,2} \right| \le K_i h^5 w_7(f;h)$ for i= 1, …, n-1

where

$e_{i,2} = 2a_{i,2} - y_i''$, K_i depend on the numbers of intervals

(4.15)

Proof :

For $y \in C^7[0,1]$ then using Taylor's expansion formula, we have:

$$y(x) = y(x_i) + (x - x_i) y'(x_i) + \frac{(x - x_i)^2}{2} y''(x_i) + ... + \frac{(x - x_i)^7}{5040} y^{(7)}(\theta_i),$$

where $x_i < \theta_i < x_{i+1}$, and similar expressions for the derivatives of $y(x)$ can be used.

Now if i=1 then from equations (3.9) and using (4.15) we obtain

$$e_{1,2} = -\frac{h^5}{240} y^{(7)}(\theta_{1,1}) + \frac{h^5}{90} y^{(7)}(\theta_{2,1}) - \frac{h^5}{120} y^{(7)}(\theta_{3,1}) + \frac{h^5}{720} y^{(7)}(\theta_{5,1}), \qquad (4.16)$$

where $x_0 < \theta_{1,1}, \theta_{2,1}, \theta_{3,1}, \theta_{5,1} < x_1$.

from equations (4.16) we get :

$\left| e_{1,2} \right| \le \frac{h^5}{80} w_7(f;h)$ so $K_1 = \frac{1}{80}$

Also if i=2 then from equations (3.9)-(3.11) and using (4.15) we obtain

$\left| e_{2,2} \right| \le \frac{197}{240} h^5 w_7(f;h)$ so $K_2 = \frac{197}{240}$

By same way in above and using the step before K_i we can show that the inequality

$\left| e_{i,2} \right| \le K_i h^5 w_7(f;h)$ for i= 1, …, n-1

This completes the proof of the Lemma 4.1.

Lemma 4.2:

Let $y \in C^7[0,1]$. Then $\left|e_{i,3}\right| \le K_i' h^5 \, w_7(f;h)$ for i= 1, …, n-1

where

$e_{i,3} = 6a_{i,3} - y_i'''$, K_i' depend on the numbers of intervals

(4.17)

Proof :

For $y \in C^7[0,1]$ then using Taylor's expansion formula, we have:

$$y(x) = y(x_i) + (x - x_i)y'(x_i) + \frac{(x - x_i)^2}{2} y''(x_i) + ... + \frac{(x - x_i)^7}{5040} y^{(7)}(\varphi_i),$$

where $x_i < \varphi_i < x_{i+1}$, and similar expressions for the derivatives of $y(x)$ can be used.

Now if i=1 then from equations (3.10) and using (4.17) we obtain

$$e_{1,3} = -\frac{h^4}{48} y^{(7)}(\varphi_{1,1}) + \frac{h^4}{24} y^{(7)}(\varphi_{2,1}) - \frac{h^4}{24} y^{(7)}(\varphi_{4,1}) + \frac{h^4}{48} y^{(7)}(\varphi_{5,1}), \qquad (4.18)$$

where $x_0 < \varphi_{1,1}, \varphi_{2,1}, \varphi_{4,1}, \varphi_{5,1} < x_1$.

from equations (4.18) we get :

$\left|e_{1,3}\right| \le \frac{h^4}{16} w_7(f;h)$ so $K_1' = \frac{1}{16}$

Also if i=2 then from equations (3.9),(3.10) and (3.12) and using (4.18) we obtain

$\left|e_{2,3}\right| \le \frac{167}{48} h^4 \, w_7(f;h)$ so $K_2' = \frac{167}{48}$

By same way in above and using the step before K_i' we can show that the inequality $\left|e_{i,3}\right| \le K_i' h^4 \, w_7(f;h)$ for i= 1, …, n-1

This completes the proof of the Lemma 4.2

Proof of Theorem 4.2:

Let $x \in [x_i, x_{i+1}]$ where i=1, 2, …, n-1.

We have from equation (2.2) and using Taylor's expansion formula we get

$$S_{i_i}^{(7)}(x) = 5040 \, a_{k,7}.$$

(4.19)

Using (4.18) and (3.15), we obtain

$$\left|S_i^{(7)}(x) - y^{(7)}(x)\right| = \left|5040 a_{i,7} - y^{(7)}(x)\right| \le 8 w_7(f;h) + 6300 K_i w_7(f;h) + 1260 K_i' w_7(f;h) \qquad (4.20)$$

From equations (3.14) and (3.15), we obtain

$$S_i^{(6)}(x) = 720 \, a_{i,6} + 5040 \, h \, a_{i,7},$$

from which we obtain

$$S_i^{(6)}(x) - y^{(6)}(x) = 720a_{i,6} + 5040ha_{i,7} - y^{(6)}(x) \qquad (4.21)$$

From equations (4.21), (3.14) and (3.15), using Taylor's series expansion on $y^{(6)}(x)$ and $y_{i+1}^{(6)}$ about $x = x_i$, we get

$$y^{(6)}(x) = y^{(6)}(x_i) + (x - x_i)y^{(7)}(x_i)$$

$$\left|S_i^{(6)}(x) - y^{(6)}(x)\right| = \left|720a_{i,6} + 5040ha_{i,7} - y^{(6)}(x_i) + (x - x_i)y^{(7)}(\varepsilon_1)\right|$$

$$\leq h\left|5040a_{i,7} - y^{(7)}(\varepsilon_1)\right| + \left|720a_{i,6} - y^{(6)}(x_i)\right| \qquad (4.22)$$

where $x_i < \varepsilon_1 < x_{i+1}$

From (3.14) and using Taylor series expansion, we get

$$\left|720a_{i,6} - y^{(6)}(x_i)\right| = h\left|3y_i^{(7)} - 3y_i^{(7)}\right| + 2880hK_i w_7(f;h) + 600hK_i' w_7(f;h)$$

$$\leq 3hw_7(f;h) + 2880hK_i w_7(f;h) + 600hK_i' w_7(f;h) \qquad (4.23)$$

From equations (4.22)and using (4.23) we obtain

$$\left|S_i^{(6)}(x) - y^{(6)}(x)\right| \leq 11hw_7(f;h) + 9180hK_i w_7(f;h) + 1860hK_i' w_7(f;h)$$

To find $\left|S_i^{(5)}(x) - y^{(5)}(x)\right|$, we need the following:

Using Taylor's series expansion on $y^{(5)}(x)$ about $x = x_i$, we get:

$$y^{(5)}(x) = y^{(5)}(x_i) + (x - x_i)y^{(6)}(x_i) + \frac{(x - x_i)^2}{2} y^{(7)}(x_i) \qquad (4.24)$$

From equations (2.2), we get:

$$S_i^{(5)}(x) = 120\, a_{i,5} + 720\, ha_{i,6} + 2520\, h^2\, a_{i,7} \qquad (4.25)$$

From equation (3.13) - (3.15), (4.24) and (4.25)

$$\left|S_i^{(5)}(x) - y^{(5)}(x)\right| = \left|120a_{i,5} + 720ha_{i,6} + 2520h^2 a_{i,7} - y^{(5)}(x)\right|$$

$$\leq \frac{4}{3}h^2 w_7(f;h) + 660h^2 K_i w_7(h) + 120h^2 K' w_7(h) \qquad (4.26)$$

By (3.2), $S_i^{(4)}(x_0) - y_i^{(4)}(x_0) = 0$, from which we obtain

$$\left|S_i^{(4)}(x) - y^{(4)}(x)\right| = \left|\int_{x_i}^{x}\left(S_i^{(5)}(t) - y^{(5)}(t)\right)dt\right| \leq \int_{x_i}^{x}\left|S_i^{(5)}(t) - y^{(5)}(t)\right|dt$$

$$\leq \int_{x_i}^{x} \frac{4h^2}{3} w_7(f;h) + 660h^2 K_i w_7(h) + 120h^2 K' w_7(h)dt$$

$$= \frac{4h^3}{3} w_7(f;h) + + 660h^3 K_i w_7(h) + 120h^3 K' w_7(h)$$

To find $\left|S_i^{(3)}(x) - y^{(3)}(x)\right|$, we need the following:

Using Taylor's series expansion on $y^{(3)}(x)$ about $x = x_i$, we get:

$$y^{(3)}(x) = y^{(3)}(x_i) + (x - x_i) y^{(4)}(x_i) + \frac{(x-x_i)^2}{2} y^{(5)}(x_i) + \frac{(x-x_i)^3}{6} y^{(6)}(x_i) + \frac{(x-x_i)^4}{24} y^{(7)}(x_i) \qquad (4.27)$$

From equations (3.13)-(3.15), (2.2) and (4.27), we get:

$$\left|S_i^{(3)}(x) - y^{(3)}(x)\right| = \Big| y_i^{(3)} + h y_i^{(4)} + 60 h^2 a_{i,5} + 120 h^3 a_{i,6} + 210 h^4 a_{i,7} - y^{(3)}(x_i) -$$

$$(x - x_i) y^{(4)}(x_i) - \frac{(x-x_i)^2}{2} y^{(5)}(x_i) - \frac{(x-x_i)^3}{6} y^{(6)}(x_i) - \frac{(x-x_i)^4}{24} y^{(7)}(\phi_1) \Big|$$

$$\le \frac{1}{16} h^4 w_7(f;h) + \frac{45}{2} h^4 K_i w_7(f;h) + \frac{7}{2} h^4 K_i' w_7(f;h) \qquad (4.28)$$

where $x_i < \phi_1 < x_{i+1}$.

Using Taylor's series expansion on $y^{(2)}(x)$ about $x = x_i$, we get:

$$y^{(2)}(x) = y^{(2)}(x_i) + (x - x_i) y^{(3)}(x_i) + \frac{(x-x_i)^2}{2} y^{(4)}(x_i) + \frac{(x-x_i)^3}{6} y^{(5)}(x_i)$$

$$+ \frac{(x-x_i)^4}{24} y^{(6)}(x_i) + \frac{(x-x_i)^5}{120} y^{(7)}(x_i) \qquad (4.29)$$

From equations (3.13)-(3.15), (2.2) and (4.29), we get:

$$\left|S_i^{(2)}(x) - y^{(2)}(x)\right| = \Big| y_i^{(2)} + h y_i^{(3)} + \frac{h^2}{2} y_i^{(4)} + 20 h^3 a_{i,5} + 30 h^4 a_{i,6} + 42 h^5 a_{i,7} - y^{(2)}(x_i)$$

$$- (x - x_i) y^{(3)}(x_i) - \frac{(x-x_i)^2}{2} y^{(4)}(x_i) - \frac{(x-x_i)^3}{6} y^{(5)}(x_i) - \frac{(x-x_i)^4}{24} y^{(6)}(x_i)$$

$$- \frac{(x-x_i)^5}{120} y^{(7)}(\lambda_2) \Big| \le \frac{1}{80} h^5 w_7(f;h) + \frac{7}{2} h^5 K_i w_7(f;h) + \frac{1}{2} h^5 K_i' w_7(f;h) \qquad (4.30)$$

From (3.2) and (4.30), we have $S_i'(x_i) - y'(x_i) = 0$, from which we obtain

$$\left|S_i'(x) - y_i'(x)\right| = \left| \int_{x_i}^{x} \left(S_i''(t) - y_i''(t)\right) dt \right| \le \int_{x_i}^{x} \left|S_i''(t) - y_i''(t)\right| dt$$

$$\le \frac{1}{80} \int_{x_i}^{x} h^5 w_7(f;h) dt + \frac{7}{2} \int_{x_i}^{x} h^5 K_i w_7(f;h) dt + \frac{1}{2} \int_{x_i}^{x} h^5 K_i' w_7(f;h) dt$$

$$= \frac{1}{80} h^6 w_7(f;h) + \frac{7}{2} h^6 K_i w_7(f;h) + \frac{1}{2} h^6 K_i' w_7(f;h) \qquad (4.31)$$

Also From (3.2) and (4.31), we have $S_i(x_i) - y(x_i) = 0$, from which we obtain

$$\left|S_i(x) - y_i(x)\right| = \left|\int_{x_i}^{x}\left(S_i'(t) - y_i'(t)\right)dt\right| \le \int_{x_i}^{x}\left|S_i'(t) - y_i'(t)\right|dt$$

$$\le \frac{1}{80}\int_{x_i}^{x} h^6 w_7(f;h)dt + \frac{7}{2}\int_{x_i}^{x} h^6 K_i w_7(f;h)dt + \frac{1}{2}\int_{x_i}^{x} h^6 K_i' w_7(f;h)dt$$

$$= \frac{1}{80}h^7 w_7(f;h) + \frac{7}{2}h^7 K_i w_7(f;h) + \frac{1}{2}h^7 K_i' w_7(f;h)$$

This proves Theorem 4.2 for $x \in [x_i, x_{i+1}],\ i = 1, 2, ..., n-1$

5. NUMERICAL RESULTS

This section presents numerical result to demonstrate the convergence of the spline function of degree seven which constructed in section 4 to the second order of initial value problem.

The out lines of seventic spline functions in this chapter are given by the following algorithm:

Algorithm SSF:

Step 1: Partition $[a,b]$ into N subintervals I.

Step 2: Set

$S_i = f_i$ (i=0, 1, 2, …, N)

$S_i''' = f_i'''$ (i=0, 1, 2, …, N-1)

$S_i^{(5)} = f_i^{(5)}$ (i=0, 1, 2, …, N-1)

and $S_0' = f_0'$.

Step 3: Find S_i, $i = 1, 2, ..., N$, from equations (3.10) to (3.12), and i=0 find s_0', $s_0^{(2)}$, $s_0^{(4)}$ and $s_0^{(6)}$.

Step 4: Find s_i', $s_i^{(2)}$, $s_i^{(4)}$ and $s_i^{(6)}$ at N equally spaced points in each subinterval $x \in [x_{i-1}, x_i]$ go to step 5, else *i*=*i*+1 and repeat this iteration to find a proper *i*.

Step 5: Stop.

Example 5.1:

We consider the second order initial value problem $y'' = \frac{1}{2}(y' + y)$ where $x \in [0,1]$ and $y(0) = y'(0) = 1$ with the exact solution $y(x) = e^x$.

Example 5.2:

Let $y'' - y = x$ where $x \in [0,1]$ and $y(0) = 1,\ y'(0) = 0$.

After solving this Example by Algorithm SSF, the following results in Table 5.1 are obtained for different step sizes *h*.
From equation (2.1) it's easy to verify that:

$$S_0(x_1) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} + h^{(5)}a_{0,5} + h^6 a_{0,6} + h^7 a_{0,7}$$

$$= y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} + h^5\left[\frac{21}{2h^5}[y_1 - y_0] - \frac{1}{2h^4}[4y_1' + 17y_0']\right.$$

$$\left.-\frac{13}{4h^3}y_0'' - \frac{3}{4h^2}y_0''' + \frac{1}{240h}\left[y_1^{(4)} - 26y_0^{(4)}\right]\right] + h^6\left[\frac{-14}{h^6}[y_1 - y_0] + \frac{1}{h^5}[3y_1' + 11y_0']\right.$$

$$\left.+\frac{4}{h^4}y_0'' + \frac{5}{6h^3}y_0''' - \frac{1}{120h^2}\left[y_1^{(4)} - 11y_0^{(4)}\right]\right] + h^7\left[\frac{9}{2h^7}[y_1 - y_0] - \frac{1}{2h^6}[2y_1' + 7y_0']\right.$$

$$\left.-\frac{5}{4h^5}y_0'' - \frac{1}{4h^4}y_0''' - \frac{1}{240h^3}\left[y_1^{(4)} - 6y_0^{(4)}\right]\right] = y_1(x).$$

Also it is easy from equations (2.1) and (2.2) to verify that:

$S_i(x_{i+1}) = y_{i+1}$ for i=0,1,…, n-1.

From (3.3.1) we have

$S_i'(x_{i+1}) = y_{i+1}'$ and $S_i^{(4)}(x_{i+1}) = y_{i+1}^{(4)}$.

From (2.1) and (2.2), with using the values of $a_{i,j}$, i=0,1,…,n-1 and j=2,3,5,6 and 7 given in the equations (3.5)-(3.12), we get

$$a_{i,2} - a_{i+1,2} + 3ha_{i,3} + 10h^3 a_{i,5} + 15h^4 a_{i,6} + 21h^5 a_{i,7} = -\frac{h^2}{4}y_i^{(4)},$$

$$a_{i,3} - a_{i+1,3} + 10h^2 a_{i,5} + 20h^3 a_{i,6} + 35h^4 a_{i,7} = -\frac{h}{6}y_i^{(4)}$$

$$S_i''(x) = -\frac{2}{h}a_{i,1} + \frac{2}{h^2}[y_{i+1} - y_i] + \frac{5h^2}{12}[y_{i+1}^{(3)} + y_i^{(3)}] + \frac{h^3}{360}[11y_{i+1}^{(5)} + 10y_i^{(5)}],$$

$$S_i^{(5)}(x) = \frac{1}{h}[y_{i+1}^{(3)} - y_i^{(3)}] + \frac{h}{6}[2y_{i+1}^{(5)} + y_i^{(5)}],$$

and

$$S_i^{(6)}(x) = \frac{1}{h}[y_{i+1}^{(5)} - y_i^{(5)}].$$

It turns out that the seven degree spline function which is presented in this chapter, yields approximate solution of order $O(h^6)$ as stated in Theorem 4.1.

After solving this Example by Algorithm SSF, the following results in Table 5.2 are obtained for different step sizes *h*.

Table 5.1 Absolute maximum error for $S(x)$ and it's derivative with different values of h for Example 5.1:

H	$\lVert s(x)-y(x)\rVert_\infty$	$\lVert s'(x)-y'(x)\rVert_\infty$	$\lVert s''(x)-y''(x)\rVert_\infty$
0.1	$2.511324481702104\times10^{-13}$	$2.009237221045623\times10^{-11}$	$1.408421601567511\times10^{-9}$
0.166	$1.504396607288072\times10^{-11}$	$7.238227794914565\times10^{-10}$	$3.049268282317996\times10^{-8}$
0.2	$6.4931837684412\times10^{-11}$	$2.604614302015307\times10^{-9}$	$9.149345103764972\times10^{-8}$

H	$\lVert s'''(x)-y'''(x)\rVert_\infty$	$\lVert s^{(4)}(x)-y^{(4)}(x)\rVert_\infty$	$\lVert s^{(5)}(x)-y^{(5)}(x)\rVert_\infty$
0.1	$8.471785895025619\times10^{-8}$	$4.251408981081895\times10^{-6}$	$1.709692787161821\times10^{-4}$
0.166	$1.102171894196147\times10^{-6}$	$3.325237181872609\times10^{-5}$	$8.048428525415652\times10^{-4}$
0.2	$2.75815930184109\times10^{-6}$	$6.94248268364727\times10^{-5}$	$14.02757113864\times10^{-4}$

H	$\lVert s^{(6)}(x)-y^{(6)}(x)\rVert_\infty$	$\lVert s^{(7)}(x)-y^{(7)}(x)\rVert_\infty$
0.1	$5.173753267333\times10^{-3}$	$1.05223817225512\times10^{-1}$
0.166	$14.693531376290\times10^{-3}$	$1.8135869829972\times10^{-1}$
0.2	$21.402760526281\times10^{-2}$	$2.21402848891596\times10^{-1}$

Table 5.2 Absolute maximum error for $S(x)$ and its derivative with different values of h for Example 5.2:

H	$\|s(x)-y(x)\|_\infty$	$\|s'(x)-y'(x)\|_\infty$	$\|s''(x)-y''(x)\|_\infty$
0.1	$2.511324481702104\times10^{-12}$	$2.009215016585131\times10^{-11}$	$1.408275940306680\times10^{-9}$
0.2	$6.493183768441213\times10^{-11}$	$2.604614274259731\times10^{-9}$	$9.149348123571599\times10^{-8}$
0.3	$1.683146066966401\times10^{-9}$	$4.507600320780014\times10^{-8}$	$1.057575988561155\times10^{-6}$

H	$\|s'''(x)-y'''(x)\|_\infty$	$\|s^{(4)}(x)-y^{(4)}(x)\|_\infty$	$\|s^{(5)}(x)-y^{(5)}(x)\|_\infty$
0.1	$8.470715817665564\times10^{-8}$	$4.251408981081895\times10^{-6}$	$1.710021012883978\times10^{-4}$
0.2	$2.758159685534167\times10^{-6}$	$6.942482683647278\times10^{-5}$	$14.02758364602\times10^{-4}$
0.3	$2.130757576601638\times10^{-5}$	$3.588075760032616\times10^{-4}$	$4.858807690758\times10^{-3}$

H	$\|s^{(6)}(x)-y^{(6)}(x)\|_\infty$	$\|s^{(7)}(x)-y^{(7)}(x)\|_\infty$
0.1	$5.175134185881\times10^{-2}$	$1.05246846504476\times10^{-1}$
0.2	$21.4027648482\times10^{-2}$	$2.21402821388476\times10^{-1}$
0.3	$4 .9858808983898\times10^{-2}$	$3.49858815133881\times10^{-1}$

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