The Class of Isomorphic Diagram Groups over Semigroup Presentations

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Abstract

In this paper, we consider the semigroup presentations of natural numbers with two and three different initial generators and union of these semigroup presentations by adding the relations. We will determine the class of diagram groups over mentioned semigroup presentations, are isomorphic to $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}$, and $\mathbb{Z} \cdot \mathbb{Z}$.

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1 Introduction

Let $P_1 = \langle x, y, z | x = y, y = z, x = z \rangle$, $P_2 = \langle a, b, c | a = b, a = c, b = c \rangle$, $P = \langle x, y, z, a, b, c | x = y, y = z, x = z, a = b, a = c, b = c \rangle$ and $P' = \langle x, y, z, a, b, c | x = y, y = z, x = z, a = b, a = c, b = c, x = a \rangle$ and
\( P^\ast = \langle x, y, z, a, b, c, t \mid x = y, y = z, x = z, a = b, a = c, b = c, xt = x, ta = a \rangle \) be

semigroup presentations. Now we consider the semigroup presentation \( P \) obtained from union of initial generators and relations of \( P_1 \) and \( P_2 \). We may obtain \( P' \) by adding a relation \( x = a \) to \( P \).

Also consider the semigroup presentation \( P^\ast \) obtained from union of initial generators and relations of \( P_1 \) and \( P_2 \) by adding relations \( xt = x, ta = a \). Using the following lemmas we may obtain the class of diagram groups are isomorphic to \( \mathbb{Z} \times \mathbb{Z} \) and \( \mathbb{Z} \cdot \mathbb{Z} \) (refer to section 4).

In section 2 we will determine the graphs \( \Gamma_{p_n} (n \in N) \) obtained from the above-mentioned semigroup presentations. We give some theorems and lemmas that class of diagram groups and the class of groups represent able by diagrams is closed under several groups theoretic constructions in section 3. By considering the theorems and lemmas in section 3, we will obtain the class of diagram groups isomorphic to \( \mathbb{Z} \times \mathbb{Z} \), and \( \mathbb{Z} \cdot \mathbb{Z} \) in section 4.

2. Determining the graphs \( \Gamma_{p_n} (n \in N) \)

Let \( P = \langle x, y, z, a, b, c \mid x = y, y = z, x = z, a = b, a = c, b = c, x = a \rangle \) be a semigroup presentation which is obtained from union of initial generators and relations of \( P_1 \) and \( P_2 \) by adding a relation \( x = a \). Associated with presentation \( P = \langle X \mid R \rangle \), we have a graph \( \Gamma \) where the vertices are words on \( X \) and the edges are the form \( e = (W_1, R_{\epsilon} \rightarrow R_{\tau}, W_2) \) such that \( \tau(e) = W_1 R_{\epsilon} W_2 \). The graph obtained from \( P \) is collections of subgraphs \( \Gamma_n \). Note that the graph \( \Gamma_{p_1} \) obtained from \( \Gamma_{p_1} \) is just a collection of subgraphs \( \Gamma_{p_1} \), where \( \Gamma_{p_1} \) contains all vertices of length \( n \) and respective edges. Similarly we obtain \( \Gamma_{p_2} \) for \( P_2 \). Now for \( P \), the graph \( \Gamma_{p_n} = \Gamma_{p_1} \cup \Gamma_{p_2} \cup \{(u, x \rightarrow a, v) \} \) such length \( uv = n - 1 \). If \( W_n \) is a vertex in \( \Gamma_{p_n} \), then \( W_n g \) \((g \in \{ x, y, z, a, b, c \})\) is a vertex in \( \Gamma_{p_{n+1}} \). Similarly if \( (u, R_{\epsilon} \rightarrow R_{-\epsilon}, v) \) is a edge in \( \Gamma_{p_n} \), then \( (u, R_{\epsilon} \rightarrow R_{-\epsilon}, vg) \) is the respective edges in \( \Gamma_{p_{n+1}} \). Thus \( \Gamma_{p_{n+1}} \) is just six copy of \( \Gamma_{p_n} \) together with six vertices \( (u, x \rightarrow a, vg) \) \((g \in \{ x, y, z, a, b, c \})\).
3. Diagram groups and group theoretic constructions

In this section we give some theorems and lemmas that class of diagram groups and the class of groups representable by diagrams is closed under several groups theoretic constructions (for detail refer to [1,3,4,5]). We can see the proof of the following lemmas and theorem in [1].

3.1 Lemma: Let \( \rho_i = \langle X_i | R_i \rangle (i = 1, 2) \) be semigroup presentations with \( X_1 \cap X_2 = \emptyset \). Let \( u_i \in X_i^* \). Let \( \rho = \langle X_1 \cup X_2 | R_1 \cup R_2 \rangle \) be the free product of \( \rho_1 \) and \( \rho_2 \). Then the diagram group \( D(\rho, u_1, u_2) \) is isomorphic to the direct product of the diagram groups \( D(\rho_1, u_1) \) and \( D(\rho_2, u_2) \).

3.2 Remark: If \( \rho = \langle X | R \rangle \) be a semigroup presentation and \( u, v \) are words over \( X \) then \( D(\rho, u) \times D(\rho, v) \) is embeddable in \( D(\rho, uv) \).
3.3 Lemma: Let $\rho_i = \langle X_i \mid R \rangle (i = 1, 2)$ be semigroup presentations and let $u_i (i = 1, 2)$ be a word over the alphabet $X_i^+$. Let $X_1$ and $X_2$ be disjoint sets. Suppose that the congruence class of $u_i$ modulo $\rho_i$ does not contain words of the form $xu_iy$ where $xu_iy$ and $xu_iy$ are words and $xu_iy$ is non empty. Consider the following presentation $\rho = \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{u_i = u_i\} \rangle$. Then $D(\rho, u_i) = D(\rho_1, u_i) \times D(\rho_2, u_i)$.

3.4 Theorem: Let $\rho = \langle X \mid R \rangle$ be a semigroup presentation, add a new generators $x_1, x_2, x_3, \ldots, x_n$ to $X$, and consider the new semigroup presentation $\rho_n = \langle X \cup \{x_1, x_2, \ldots, x_n\} \mid R \rangle$. Then for every word $u$ over $X$ the diagram groups $D(\rho, u), D(\rho_1, xu_1), D(\rho_2, x_2xu_1x_2), \ldots, D(\rho_n, x_nx_nu_nx_nx_n)$ are isomorphic.

Proof: Let $\rho = \langle X \mid R \rangle$ be a semigroup presentation. Add a new generator $x_1 \in X$, and consider $\rho_1 = \langle X \cup \{x_1\} \mid R \rangle$. Then the $D(\rho, u)$ and $D(\rho_1, xu_1)$ are isomorphic. The isomorphic between these two diagram groups is induced by the map $\Delta \to \varepsilon(x_1) + \Delta + \varepsilon(x_1)$ where $\Delta$ is a diagram group over $\rho$.

Every word in the congruence class $x_1u_1x_1$ of modulo $\rho_1$ has the form $x_1v_1x_1$ where $v$ is a word from the congruence class of $u$ modulo $\rho$. Thus $x_1u_1x_1$ satisfies the condition of lemma that is if $x_1u_1x_1 = s$ modulo $\rho_1$ then $st = \phi$.

Now if we add a new generator $x_2 \in X$ and consider the presentation $\rho_2 = \langle X \cup \{x_1, x_2\} \mid R \rangle$. Then for every word $s$ over $X \cup \{x_1\}$, the diagram group $D(\rho_1, xu_1x_1) \equiv D(\rho_2, x_2x_1u_1x_2x_1)$. $s \in X \cup \{x_1\} \Rightarrow \exists$ word $v$ over $X$ that's $s = x_1v_1x_1$, $u \equiv v$ (modulo $\rho_1$), then $D(\rho_1, xu_1x_1) \equiv D(\rho_2, x_2x_1u_1x_2x_1)$.

The isomorphic between these two diagram groups is induced by the map $\Delta_1 \to \varepsilon(x_2) + \Delta_1 + \varepsilon(x_2)$ where $\Delta_1$ is a diagram group over $\rho_1$.

Thus as above method we may obtained

$$D(\rho_{n-1}, x_{n-1} \ldots x_2x_1u_1x_2x_3 \ldots x_{n-1}) \equiv D(\rho_n, x_n \ldots x_2x_1u_1x_2 \ldots x_3 \ldots x_n).$$

The isomorphic between these two diagram groups is induced by the map $\Delta_{n-1} \to \varepsilon(x_n) + \Delta_{n-1} + \varepsilon(x_n)$ where $\Delta_{n-1}$ is a diagram group over $\rho_{n-1}$.

3.5 Lemma: Let $\rho_i = \langle X_i \mid R \rangle (i = 1, 2)$ be semigroup presentations and let $u_i (i = 1, 2)$ be a word over the alphabet $X_i^+$. Let $X_1$ and $X_2$ be disjoint sets. Suppose that the congruence class of $u_i$ modulo $\rho_i$ does not contain words of the form $xu_iy$ where $xu_iy$ and $xu_iy$ are words and $xu_iy$ is non empty. Consider the following
presentation $\rho = \langle X_1 \cup X_2 \cup \{c\} | R_1 \cup R_2 \cup \{u,c = u_1, cu_2 = u_2\} \rangle$. Then $D(\rho, u_1, u_2) \cong D(\rho_1, u_1) \ast D(\rho_2, u_2)$. 

4. The class of isomorphic diagram groups

Kilibarda in [2], proved that the diagram group $D(P, u)$ is isomorphic to the fundamental group $\Pi_1(K(P), u)$ of the 2-complex with the base point $u$.

Guba and Sapir in [1], have shown $D(P_1, x), D(P_2, a)$ those are infinite cyclic. Because the fundamental groups of $P_1$ and $P_2$ have one generator in its graphs, and the fundamental groups of $P_1$ and $P_2$ are free of rank 1. Then $D(P_1, x), D(P_2, a)$ are isomorphic to $Z$.

Now by considering the theorems and lemmas in section 3, we may obtain the class of diagram groups isomorphic to $Z \times Z, Z \ast Z, Z \ast Z$. Also if we let the semigroup presentation $S = \langle x, y | x = y \rangle$ be a natural numbers semigroup presentation with two initial generators. We will prove that $D(S, xy) \cong D(S, yx) \cong D(S, x^2) \cong D(S, y^2) \cong Z$.

Consider the following construction. Let $G, H$ be groups and $Z = \langle z \rangle$ let be infinite cyclic group. Take the free product $G \times H \times Z$ subject to the following relations $[g^n, h] = 1$ for all $g \in G, h \in H, n = 0,1,2,\ldots$. Let us denote the resulting group by $G \ast H$. This construction has several nice properties:

1) The natural homomorphism of $G$ and $H$ into $G \ast H$ are embedding $G$ and $H$ are retracts of $G \ast H$. Indeed the homomorphism $g \rightarrow g, h \rightarrow h, z \rightarrow 1$ where $g \in G, h \in H$ is a retraction of $G \ast H$ onto $G$, the homomorphism $g \rightarrow 1, h \rightarrow h, z \rightarrow 1$ is a retraction of $G \ast H$ onto $G \ast H$.

2) $G \ast H$ and $H \ast G$ are isomorphic. Indeed, the map $g \rightarrow g, h \rightarrow h, z \rightarrow z^{-1}$ can be extended to an isomorphism from $G \ast H$ to $H$.

4.1 Lemma: $D(P_1, uv) \cong D(P_2, rs) \cong Z \times Z$, for $\forall u, v \in \{x, y, z\}$ and for $\forall r, s \in \{a, b, c\}$.

Proof. By considering the $D(P_1, u) = Z, u \in \{x, y, z\}$ and $D(P_2, v) = Z, v \in \{a, b, c\}$, and remark 3.2, $D(P_1, uv) \cong D(P_1, u) \times D(P_1, v) \cong Z \times Z$, and $D(P_2, rs) \cong D(P_2, r) \times D(P_2, s) \cong Z \times Z$. 


4.2 Lemma: Let $P = \langle x, y, z \mid x = y, y = z, x = z \rangle$ be a semigroup presentation. By adding the new generators $x_1, x_2, x_3, \ldots, x_n$ to $\{x, y, z\}$, and consider the new semigroup presentation $\rho_n = \langle \{x, y, z\} \cup \{x_1, x_2, \ldots, x_n\} \mid R \rangle$. Then for every word $u$ over $\{x, y, z\}$ the diagram groups $D(\rho, u), D(\rho_1, u_1), D(\rho_2, x_1 u x_2), D(\rho_n, x_1 x_2 x_3 \ldots x_n)$ are isomorphic to $\mathbb{Z}$.

Proof: The result follows immediately from Theorems 3.4 and $D(P, u) = \mathbb{Z}, u \in \{x, y, z\}$.

4.3 Lemma: Let $P_1 = \langle x, y, z \mid x = y, y = z, x = z \rangle$, $P_2 = \langle a, b, c \mid a = b, a = c, b = c \rangle$, and $P' = \langle x, y, z, a, b, c, t \mid x = y, y = z, x = z, a = b, a = c, b = c, xt = x, ta = a \rangle$ be semigroup presentations. Then $D(\rho', u_1, u_2) \cong D(\rho_1, u_1) \cdot D(\rho_2, u_2) = \mathbb{Z} \cdot \mathbb{Z}$.

Proof: The lemma is an immediate consequence of $D(P, u) = \mathbb{Z}, u \in \{x, y, z\}$ and lemma 3.5.

4.4 Lemma: Let $P_1 = \langle x, y, z \mid x = y, y = z, x = z \rangle$, $P_2 = \langle a, b, c \mid a = b, a = c, b = c \rangle$, and $P = \langle x, y, z, a, b, c \mid x = y, y = z, x = z, a = b, a = c, b = c \rangle$ be semigroup presentations. Then the diagram group $D(\rho, u_1, u_2)$ is isomorphic to the direct product of the diagram groups $D(\rho_1, u_1)$ and $D(\rho_2, u_2)$ that is isomorphic to $\mathbb{Z} \times \mathbb{Z}, u_1, u_2 \in \{x, y, z\}$.

Proof: The lemma is an immediate consequence of $D(P, u) = \mathbb{Z}, u \in \{x, y, z\}$ and lemma 3.1.

4.5 Lemma: Let $S = \langle x, y \mid x = y \rangle$ be a natural numbers semigroup presentation with two initial generators. Then $D(S, xy) \cong D(S, yx) \cong D(S, x^2) \cong D(S, y^2) \cong \mathbb{Z}$.

Proof: Let $S = \langle x, y \mid x = y \rangle$ be a semigroup presentation. Then the graph of $\Gamma(S)$ is (in figure 2).
Class of isomorphic diagram groups

The graph of $\Gamma_2(S)$ is just two copy of $\Gamma_1(S)$ and each vertex in each copy are joined together respectively (see figure (3)).

![Graph Γ1(S)](image1)

Figure (2) : Graph $\Gamma_1(S)$

The spanning tree of $\Gamma_2(S)$ is (in figure (4)), so that fundamental group has a one generator in graph $\Gamma_2(S)$, and $\Pi_1(K(P), x^2)$ is a free group one rank 1.

![Spanning tree of graph Γ2(S)](image2)

Figure (3) : Graph $\Gamma_2(S)$

Hence $D(S, x^2) \cong Z$. Similarly we can conclude that $D(S, xy) \cong D(S, yx) \cong D(S, y^2) \cong Z$.

References


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