On Approximation of Conjugate of a Function

Belonging to \( LIP(\xi(t), r) \) Class by Product Summability

Means of Conjugate Series of Fourier Series

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Abstract

In this paper, a new theorem on degree of approximation of conjugate of a function \( f \in Lip(\xi(t), r) \) class using \((E,q)(C,1)\) product summability means of conjugate series of Fourier series has been established.

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1. Introduction

Alexits[5], Sahney and Goel[3], Chandra[13], Qureshi and Neha[8], Liendler[11] and Rhoades[2] have determined the degree of approximation of a function belonging to \( Lip(\alpha) \) class by Cesáro, Nörlund and generalized Nörlund single summability methods. Working in the same direction Sahney & Rao[4] and Khan[6] have studied the degree of approximation of a function belonging to \( Lip(\alpha, r) \) class by Nörlund and generalized Nörlund means. Thereafter Qureshi[9,10] discussed the degree of approximation of conjugate of a function belonging to \( Lip(\alpha) \) class and \( Lip(\alpha, r) \) class by Nörlund means of conjugate series of Fourier series. But nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function belonging to \( Lip(\xi(t), r) \) class by \((E,q)(C,1)\) product summability means. The \( Lip(\xi(t), r) \) class is a generalization of \( Lip(\alpha) \) class and \( Lip(\alpha, r) \) class. Therefore, in present paper, a theorem on degree of approximation of conjugate of
a function belonging to $Lip(\xi(t), r)$ class by $(E,q)(C, 1)$ product summability means of conjugate series of Fourier series has been proved.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its $n^{th}$ partial sums $\{s_n\}$. The $(E,q)$ transform is defined as the $n^{th}$ partial sum of $(E,q)$ summability and we denote it by $E_n^q$. If

$$E_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \to s \text{ as } n \to \infty \quad (1.1)$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable $(E,q)$ to a definite number $s$ [6].

If

$$t_n = \frac{s_0 + s_1 + s_2 + \ldots + s_n}{n+1}$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s \text{ as } n \to \infty \quad (1.2)$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number $s$ by $(C, 1)$ method. The $(E,q)$ transform of the $(C, 1)$ transform defines $(E,q)(C, 1)$ product transform and we denote it by $E_n^q C_n^1$. Thus if

$$E_n^q C_n^1 = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} C_k^1 \to s \text{ as } n \to \infty \quad (1.3)$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E,q)(C, 1)$ method or summable $(E,q)(C, 1)$ to a definite number $s$.

Let $f(x)$ be a $2\pi$ - periodic function and Lebesgue integrable. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1.4)$$

with $n^{th}$ partial sum $s_n(f;x)$. The conjugate series of Fourier series (1.4) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \quad (1.5)$$

A function $f \in Lip \alpha$, if

$$f(x+t) - f(x) = O \left( \left| t^\alpha \right| \right) \text{ for } 0 < \alpha \leq 1 \quad (1.6)$$

$f \in Lip (\alpha, r)$, if

$$\left( \int_0^{2\pi} |f(x+t) - f(x)|^r \, dx \right)^{\frac{1}{r}} = O \left( \left| t^\alpha \right| \right), 0 < \alpha \leq 1, r \geq 1 \quad (1.7)$$

(Definition 5.38 of Mc Fadden [12])
Approximation of conjugate of a function

and that $f \in \text{Lip } (\xi(t),r)$, if

$$\left( \int_{0}^{2\pi} |f(x + t) - f(x)|^r \, dx \right)^{\frac{1}{r}} = O(\xi(t)), \quad r \geq 1,$$

(1.8)

where $\xi(t)$ is a positive increasing function of $t$.

If $\xi(t) = t^{\alpha}$ then $\text{Lip}(\xi(t),r)$ class coincides with the class $\text{Lip}(\alpha,r)$ and if $r \to \infty$ then $\text{Lip}(\alpha,r)$ class reduces to the class $\text{Lip } \alpha$.

$L_r$ - norm is defined by

$$\|f\|_r = \left( \int_{0}^{2\pi} |f(x)|^r \, dx \right)^{\frac{1}{r}}, \quad r \geq 1$$

(1.9)

and let the degree of approximation be given by (Zygmund[1])

$$E_n(f) = \min_{\psi} \|f - f\|_r$$

(1.10)

where $t_n(x)$ is some $n^{th}$ degree trigonometric polynomial.

We use the following notations throughout this paper

$$\psi(t) = f(x + t) - f(x-t)$$

$$\overline{G}_n(t) = \frac{1}{2\pi(1 + q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{1}{(1 + k)} \sum_{\nu=0}^{k} \cos \left( \nu + \frac{1}{2} \right) t = O \left( \frac{1}{n+1} \right)$$

(2.1)

provided $\xi(t)$ satisfies the following conditions:

$$\left( \int_{0}^{1} \left( \frac{t}{\xi(t)} \right)^r \, dt \right)^{\frac{1}{r}} = O \left( \frac{1}{n+1} \right)$$

(2.2)

2. Main Theorem

We prove the following theorems:

**Theorem:** If $\overline{f}(x)$, conjugate to a $2\pi$-periodic function $f$ belonging to $\text{Lip}(\xi(t),r)$ class, then its degree of approximation by $(E,q)(C,1)$ product summability means of conjugate series of Fourier series is given by

$$\|E^q C^1_n - \overline{f}\|_r = O \left( \frac{1}{n+1} \right)^{\frac{1}{r}} \xi \left( \frac{1}{n+1} \right)$$

(2.1)

provided $\xi(t)$ satisfies the following conditions:
and

\[
\left\{ \int_1^{\frac{1}{r}} \left( \frac{t^{-\delta}|\psi(t)|}{\xi(t)} \right)^{\frac{1}{r}} \right\} = O \left( (n+1)^{\frac{1}{r}} \right)
\]

(2.3)

where \( \delta \) is an arbitrary number such that \( s(1 - \delta) - 1 > 0 \), \( \frac{1}{r} + \frac{1}{s} = 1 \), \( 1 \leq r \leq \infty \), conditions (2.2) and (2.3) hold uniformly in \( x \), \( E_n^s C_n^1 \) is \((E, q)(C, 1)\) means of the series (1.5) and

\[
\overline{f}(x) = \frac{1}{2\pi} \int_0^{2\pi} \psi(t) \cot \left( \frac{t}{2} \right) dt
\]

(2.4)

3. Lemmas

For the proof of our theorem, following lemmas are required:

**Lemma 1.** \( \overline{G_n(t)} = O \left[ \frac{1}{t} \right] \) for \( 0 \leq t \leq \frac{1}{n+1} \)

**Proof.** For \( 0 \leq t \leq \frac{1}{n+1} \), \( \sin \left( \frac{t}{2} \right) \geq \frac{t}{\pi} \) and \( |\cos nt| \leq 1 \)

\[
\left| \overline{G_n(t)} \right| = \frac{1}{2\pi(1+q)^n} \left\{ \sum_{k=0}^{n} \frac{n}{k} q^{n-k} \left( \frac{1}{1+k} \right) \sum_{\nu=0}^{k} \frac{\cos \left( \nu + \frac{1}{2} \right)}{\sin \left( \frac{t}{2} \right)} \right\}
\]

\[
\leq \frac{1}{2\pi(1+q)^n} \left\{ \sum_{k=0}^{n} \frac{n}{k} q^{n-k} \left( \frac{1}{1+k} \right) \sum_{\nu=0}^{k} \frac{\cos \left( \nu + \frac{1}{2} \right)}{\sin \left( \frac{t}{2} \right)} \right\}
\]

\[
\leq \frac{1}{2(1+q)^n} \left\{ \sum_{k=0}^{n} \frac{n}{k} q^{n-k} \left( \frac{1}{1+k} \right) \sum_{\nu=0}^{k} (1) \right\}
\]

\[
= \frac{1}{2(1+q)^n} \left\{ \sum_{k=0}^{n} \frac{n}{k} q^{n-k} \left( \frac{1}{1+k} \right) \sum_{\nu=0}^{k} (1) \right\}
\]

\[
= \frac{1}{2(1+q)^n} (1+q)^n
\]

\[
= O \left[ \frac{1}{t} \right] \quad \text{since} \quad \sum_{k=0}^{n} \frac{n}{k} q^{n-k} = (1+q)^n
\]
Lemma 2: \( \left| G_n(t) \right| = O\left( \frac{1}{t} \right) \), for 0 ≤ a ≤ b ≤ ∞, 0 ≤ t ≤ π and any n

Proof. For 0 < \( \frac{1}{n+1} \) ≤ t ≤ π, \( \sin \frac{t}{2} \geq \frac{t}{\pi} \)

\[
\left| G_n(t) \right| = \frac{1}{2\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{1+k} \right) \left( \sum_{v=0}^{k} \frac{\cos\left( v + \frac{1}{2} \right) t}{\sin \frac{t}{2}} \right) \right|
\]

\[
\leq \frac{1}{2\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{1+k} \right) \Re \left\{ \sum_{v=0}^{k} e^{ivt} \right\} \right| e^{\frac{t}{2}}
\]

\[
\leq \frac{1}{2\pi(1+q)^n} \left| \sum_{k=0}^{n-1} \binom{n}{k} q^{-k} \left( \frac{1}{1+k} \right) \Re \left\{ \sum_{v=0}^{k} e^{ivt} \right\} \right|
\]

\[
+ \frac{1}{2\pi(1+q)^n} \left| \sum_{k=n}^{n-1} \binom{n}{k} q^{-k} \left( \frac{1}{1+k} \right) \Re \left\{ \sum_{v=0}^{k} e^{ivt} \right\} \right| \quad \text{(3.1)}
\]

Now considering first term of (3.1)

\[
\frac{1}{2\pi(1+q)^n} \left| \sum_{k=0}^{n-1} \binom{n}{k} q^{-k} \left( \frac{1}{1+k} \right) \Re \left\{ \sum_{v=0}^{k} e^{ivt} \right\} \right| \leq \frac{1}{2\pi(1+q)^n} \left| \sum_{k=0}^{n-1} \binom{n}{k} q^{-k} \left( \frac{1}{1+k} \right) \left( \sum_{v=0}^{k} 1 \right) \right| \left| e^{ivt} \right|
\]

\[
\leq \frac{1}{2\pi(1+q)^n} \left| \sum_{k=0}^{n-1} \binom{n}{k} q^{-k} \right| \quad \text{(3.2)}
\]

Now considering second term of (3.1) and using Abel’s lemma

\[
\frac{1}{2\pi(1+q)^n} \left| \sum_{k=n}^{n-1} \binom{n}{k} q^{-k} \left( \frac{1}{1+k} \right) \Re \left\{ \sum_{v=0}^{k} e^{ivt} \right\} \right| \leq \frac{1}{2\pi(1+q)^n} \frac{1}{\max_{0 \leq m \leq k} \sum_{v=0}^{m} e^{ivt}}
\]

\[
\leq \frac{1}{2\pi(1+q)^n} \sum_{k=n}^{n-1} \binom{n}{k} q^{-k} \left( \frac{1}{1+k} \right) (1+k)
\]

\[
= \frac{1}{2\pi(1+q)^n} \sum_{k=n}^{n-1} \binom{n}{k} q^{-k} \quad \text{(3.3)}
\]
Combining (3.1), (3.2) and (3.3)

\[ |G_n(t)| \leq \frac{1}{2t(1+q)^\nu} \sum_{k=0}^{n-1} \binom{n}{k} q^{-k} + \frac{1}{2t(1+q)^\nu} \sum_{k=1}^{n} \binom{n}{k} q^{-k} \]

\[ |G_n(t)| = O\left(\frac{1}{t}\right) \]

4. Proof of Theorem

Let \( s_n(x) \) denotes the partial sum of series (1.5). Then following Lal [14], we have

\[ s_n(x) - f(x) = \frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \frac{\cos\left(n + \frac{1}{2} \right) t}{\sin\left(\frac{t}{2}\right)} \, dt \]

Therefore using (1.5) the \((C,1)\) transform \( C_n^1 \) of \( s_n \) is given by

\[ C_n^1 - f(x) = \frac{1}{2\pi(n+1)} \int_{0}^{\pi} \psi(t) \sum_{k=0}^{n} \frac{\cos\left(k + \frac{1}{2} \right) t}{\sin\left(\frac{t}{2}\right)} \, dt \]

Now denoting \((E,q)(C,1)\) transform of \( s_n \) as \( E_q^1 \), we write

\[ E_q^1 C_n^1 - f(x) = \frac{1}{2\pi(1+q)^\nu} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \int_{0}^{\pi} \psi(t) \sum_{\nu=0}^{k} \frac{1}{\nu + k + 1} \cos\left(\nu + \frac{1}{2} \right) t \, dt \]

\[ = \int_{0}^{\pi} \psi(t) \, G_n(t) \, dt \]

\[ = \left[ \int_{0}^{\pi} + \int_{\frac{\pi}{n+1}}^{\pi} \right] \psi(t) \, G_n(t) \]

\[ = I_1 + I_2 \quad \text{(say)} \quad (4.1) \]

We consider,

\[ |I_1| \leq \int_{0}^{\pi} |\psi(t)| |G_n(t)| \, dt \]
Using Hölder’s inequality and the fact that $\psi(t) \in \text{Lip}(\xi(t), r)$,

$$ |I| \leq \left( \frac{1}{n+1} \int_0^1 \left\{ \frac{t|\psi(t)|}{\xi(t)} \right\}^\alpha dt \right)^{\frac{1}{\alpha}} \left[ \frac{1}{n+1} \int_0^1 \left\{ \frac{\xi(t)|G_n(t)|}{t} \right\}^\beta dt \right]^{\frac{1}{\beta}}$$

$$ = O\left( \frac{1}{n+1} \int_0^1 \left\{ \frac{\xi(t)|G_n(t)|}{t^2} \right\}^\frac{s}{2} dt \right)^{\frac{1}{\beta}} \text{ by (2.2)}$$

$$ = O\left( \frac{1}{n+1} \int_0^1 \left\{ \frac{\xi(t)}{t^2} \right\}^\frac{s}{2} dt \right)^{\frac{1}{\beta}} \text{ by Lemma 1}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$ I_1 = O\left( \frac{1}{n+1} \xi(t) \left( \frac{1}{n+1} \right) \left\{ \frac{1}{n+1} \int_0^1 \left\{ \frac{dt}{t^{2s}} \right\}^\frac{1}{\beta} \right\}^\frac{1}{2} \right) \text{ for some } 0 < \varepsilon < \frac{1}{n+1}$$

$$ = O\left( \frac{1}{n+1} \xi(t) \left( \frac{1}{n+1} \right) \left\{ \frac{1}{n+1} \int_0^1 \left\{ \frac{t^{-2s+1}}{t} \right\}^\frac{1}{\beta} \right\}^\frac{1}{\beta} \right)$$

$$ = O\left( \frac{1}{n+1} \xi(t) \left( \frac{1}{n+1} \right) \left( n+1 \right)^{-\frac{1}{2s}} \right)$$

$$ = O\left( \xi(t) \left( \frac{1}{n+1} \right)^{\frac{1}{2s}} \right)$$

$$ = O\left( (n+1)^{\frac{1}{\beta}} \xi(t) \left( \frac{1}{n+1} \right)^{\frac{1}{2s}} \right)$$

$$ \therefore \frac{1}{r} + \frac{1}{s} = 1, \ 1 \leq r \leq \infty \quad (4.2)$$

Now we consider,

$$ |I_2| \leq \frac{1}{n+1} \int \frac{\psi(t)}{G_n(t)} dt $$
Using Hölder’s inequality

\[
|I_2| \leq \left\{ \frac{\int \left( t^{-\delta} |\psi(t)| \right)^{\frac{1}{s}} dt}{n+1} \right\} \left\{ \frac{\int \left( \frac{\xi(t)G_n(t)}{t^{\delta}} \right)^{\frac{1}{s}} dt}{n+1} \right\}^{\frac{1}{s}}
\]

\[
= O \left\{ (n+1)^{\frac{\delta}{s}} \right\} \left\{ \frac{\int \left( \frac{\xi(t)G_n(t)}{t^{\delta}} \right)^{\frac{1}{s}} dt}{n+1} \right\}^{\frac{1}{s}} \quad \text{by (2.3)}
\]

\[
= O \left\{ (n+1)^{\frac{\delta}{s}} \right\} \left\{ \frac{\int \left( \frac{\xi(t)}{t^{\delta}} \right)^{\frac{1}{s}} dt}{n+1} \right\}^{\frac{1}{s}} \quad \text{by Lemma 2}
\]

Now putting \( t = \frac{1}{y} \),

\[
I_2 = O \left\{ (n+1)^{\frac{\delta}{s}} \right\} \left\{ \frac{\int \left( \frac{1}{y^{\delta-1}} \right)^{\frac{1}{s}} dy}{n+1} \right\}^{\frac{1}{s}} \frac{dy}{y^2}
\]

Since \( \xi(t) \) is a positive increasing function and using second mean value theorem for integrals,

\[
I_2 = O \left\{ (n+1)^{\frac{\delta}{s}} \xi \frac{1}{n+1} \right\} \left\{ \int_{\eta}^{\frac{1}{\eta}} \left( \frac{dy}{y^{\eta-1+2}} \right)^{\frac{1}{s}} \right\}^{\frac{1}{s}} \quad \text{for some} \quad \frac{1}{n} \leq \eta \leq n+1
\]

\[
= O \left\{ (n+1)^{\frac{\delta}{s}} \xi \frac{1}{n+1} \right\} \left\{ \int_{\eta}^{\frac{1}{\eta}} \left( \frac{dy}{y^{\eta-1+2}} \right)^{\frac{1}{s}} \right\}^{\frac{1}{s}} \quad \text{for some} \quad \frac{1}{n} \leq 1 \leq n+1
\]

\[
= O \left\{ (n+1)^{\frac{\delta}{s}} \xi \frac{1}{n+1} \right\} \left\{ \frac{y^{(\delta-1)+2}}{s(1-\delta)-1} \right\} \left\{ \frac{1}{n+1} \right\}^{\frac{1}{s}} \frac{1}{s}
\]

\[
= O \left\{ (n+1)^{\frac{\delta}{s}} \xi \frac{1}{n+1} \right\} \left\{ (n+1)^{1-\delta} \right\}
\]

\[
= O \left\{ (n+1)^{\frac{1}{r}} \xi \frac{1}{n+1} \right\} \left\{ \frac{1}{r} + \frac{1}{s} = 1 \right\}
\]

(4.3)
Combining from (4.1) to (4.3), we get
\[ \left| E_n^q C_n^i - \tilde{f} \right| = O \left( (n+1)^{\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right) \]

Now using \( L_r \)-norm, we get
\[ \left\| E_n^q C_n^i - \tilde{f} \right\| = \left\{ 2 \pi \int_0^\pi \left| E_n^q C_n^i - f(x) \right|^r dx \right\}^{\frac{1}{r}} \]
\[ = O \left[ \int_0^\pi \left( n+1 \right)^{\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right]^r dx \]
\[ = O \left( \left( n+1 \right)^{\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right)^r \]
\[ = O \left( \left( n+1 \right)^{\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right) \]

This completes the proof of theorem.

5. Applications

Following corollaries can be derived from our main theorem:

**Corollary 1:** If \( \xi(t) = t^\alpha, \ 0 < \alpha \leq 1 \), then the class \( \text{Lip}(\xi(t), r) \), \( r \geq 1 \) reduces to the class \( \text{Lip}(\alpha, r) \) and the degree of approximation of a function \( \tilde{f} \), conjugate to \( 2\pi \)-periodic function \( f \in \text{Lip}(\alpha, r), \frac{1}{r} < \alpha < 1 \), is given by
\[ \left| E_n^q C_n^i - f \right| = O \left( \frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right) \]

**Proof:** We have
\[ \left\| E_n^q C_n^i - \tilde{f} \right\| = O \left[ \frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right]^r \]
or
\[ \left\{ (n+1)^{\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right\} = O \left[ \frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right]^r \]
or
\[ O(1) = O \left( \frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right)^r \]

or
\[ O(1) = O \left( \frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right)^r = O \left( \frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right) \]
Hence
\[
\left\| E_n^q C_n^t - f \right\| = O\left( (n+1)^{\frac{1}{r}} \left( \frac{1}{n+1} \right) \right)
\]
for if not the right-hand side will be \(O(1)\), therefore
\[
\left| E_n^q C_n^t - f \right| = O\left( \left( \frac{1}{n+1} \right)^{\frac{1}{r}} (n+1)^{\frac{1}{r}} \right)
\]
\[
= O\left( \frac{1}{(n+1)^{\frac{1}{r}}} \right)
\]

**Corollary 2:** If \( r \to \infty \) in Corollary 1, then the class \( \text{Lip}(\alpha, r) \) reduces to the Class \( \text{Lip} \alpha \) and the degree of approximation of \( \overline{f} \), conjugate of a \( 2\pi \)–periodic function \( f \in \text{Lip} \alpha \), \( 0 < \alpha < 1 \) is given by
\[
\left\| E_n^q C_n^t - \overline{f} \right\| = O\left( \frac{1}{(n+1)^{\frac{1}{r}}} \right)
\]

**Corollary 3:** If \( \xi(t) = t^\alpha \), \( 0 < \alpha \leq 1 \), then the class \( \text{Lip}(\xi(t), r), r \geq 1 \), reduces to the class \( \text{Lip}(\alpha, r) \) and if \( q = 1 \) then \((E,q)\) summability reduces to \((E,1)\) summability and the degree of approximation of function \( \overline{f} \), conjugate to a \( 2\pi \)–periodic function \( f \in \text{Lip} (\alpha, r) \), \( \frac{1}{r} \leq \alpha \leq 1 \) is given by
\[
\left| E_n^q C_n^t - f \right| = O\left( \frac{1}{(n+1)^{\frac{1}{r}}} \right)
\]

**Corollary 4:** If \( r \to \infty \) in Corollary 3, then class \( \text{Lip}(\alpha, r) \) reduces to the class \( f \in \text{Lip} \alpha \) and the degree of approximation of function \( \overline{f} \), conjugate of a \( 2\pi \)–periodic function \( f \in \text{Lip} \alpha \), \( 0 < \alpha < 1 \) is given by
\[
\left\| (EC)_n^q - \overline{f} \right\| = O\left( \frac{1}{(n+1)^{\frac{1}{r}}} \right)
\]

**Remark:** Independent proofs of above corollaries 1 and 3 can be obtained along the same lines of our theorem.
References


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