Anti–Homomorphisms in Fuzzy Ideals of Rings

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Abstract

In this paper, a new concept of anti-homomorphism between two fuzzy rings $R$ and $R'$ is defined and many results analogous to homomorphism of rings are established.

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1. INTRODUCTION

2. PRELIMINARIES

2.1 Definition:
Let $X$ be a non-empty universal set. A fuzzy subset $A$ of $X$ is a function $A : X \rightarrow [0,1]$.

2.2 Definition:
A fuzzy set $\mu$ of a ring $R$ is called a fuzzy sub ring of $R$ if for all $x, y \in R$,

\[
\mu(x - y) \geq \min \{ \mu(x), \mu(y) \}
\]
\[
\mu(xy) \geq \min \{ \mu(x), \mu(y) \}
\]

2.3 Definition:
A fuzzy set $\mu$ of a ring $R$ is called a fuzzy ideal of $R$ if for all $x, y \in R$,

\[
\mu(x - y) \geq \min \{ \mu(x), \mu(y) \}
\]
\[
\mu(xy) \geq \max \{ \mu(x), \mu(y) \}
\]

2.4 Definition:
A fuzzy ideal $\mu$ of a ring $R$ is called a fuzzy maximal if $\text{Im}(\mu) = \{1, \alpha\}$ where $\alpha \in [0,1)$ and the ideal $\{ x \in R / \mu(x) = 1 \}$ is maximal.

2.5 Definition:
A fuzzy ideal $\mu$ of a ring $R$ is called a fuzzy prime if for any two fuzzy ideals $\sigma$ and $\theta$ of $R$ the condition $\sigma \theta \subseteq \mu$ implies that $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

2.6 Definition:
A fuzzy ideal $\mu$ of a ring $R$ is called a fuzzy primary if for any two fuzzy ideals $\sigma$ and $\theta$ of $R$ the conditions $\sigma \theta \subseteq \sqrt{\mu}$ and $\sigma \nsubseteq \mu$ together imply that $\theta \subseteq \sqrt{\mu}$.

2.7 Definition:
Let $f : R \rightarrow R'$ be any function, a fuzzy set $\mu$ of $R$ is called $f$-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, $x, y \in R$.

2.8 Definition:
Let $R$ and $R'$ be two rings, a mapping $f : R \rightarrow R'$ is called a fuzzy anti-homomorphism if $f(\mu + \sigma) = f(\mu) + f(\sigma)$ and $f(\mu \sigma) = f(\sigma)f(\mu)$.

2.9 Remarks:
For a fuzzy maximal ideal $\mu$ of a ring $R$, we have (i) $\mu$ is fuzzy prime and (ii) $\sqrt{\mu} = \mu$. 
3. SOME PROPOSITION

3.1 Proposition:
The anti-homomorphic image of a fuzzy ideal of $R$ is a fuzzy ideal of $R^l$.

3.2 Proposition:
The anti-homomorphic pre-image of a fuzzy ideal of $R^l$ is a fuzzy ideal of $R$.

3.3 Proposition:
Let $f : R \rightarrow R^l$ be a surjective anti-homomorphism, Let $\mu^l$ be a fuzzy prime ideal of $R^l$, then $f^{-1}(\mu^l)$ is a fuzzy prime ideal of $G$.

Proof. Let $\mu$ and $\sigma$ be any two fuzzy ideals of $R$, such that $\mu \sigma \subset f^{-1}(\mu^l)$
This implies that
\[ f(\mu \sigma) \subset ff^{-1}(\mu^l) = \mu^l \]
\[ \Rightarrow f(\sigma)f(\mu) \subset = \mu^l \text{ because } f \text{ is an anti-homomorphism} \]
\[ \Rightarrow f(\sigma) \subset \mu^l \text{ or } f(\mu) \subset \mu^l \text{ because } \mu^l \text{ is a fuzzy prime ideal of } R^l \]
\[ \Rightarrow f^{-1}(f(\sigma)) \subset f^{-1}(\mu^l) \text{ or } f^{-1}(f(\mu)) \subset f^{-1}(\mu^l) \]
\[ \Rightarrow \sigma \subset f^{-1}(\mu^l) \text{ or } \mu \subset f^{-1}(\mu^l) \]
\[ \Rightarrow f^{-1}(\mu^l) \text{ is a fuzzy prime ideal of } R^l \]

3.4 Proposition:
Let $f : R \rightarrow R^l$ be an anti-homomorphism. Let $\mu$ be any $f$-invariant fuzzy prime ideal of $R$, then $f(\mu)$ is a fuzzy prime ideal of $R^l$.

Proof. Let $\sigma^l$ and $\theta^l$ be any two fuzzy ideals of $R$, such that $\sigma^l \theta^l \subset f(\mu)$
\[ \Rightarrow f^{-1}(\sigma^l \theta^l) \subset f^{-1}(f(\mu)) = \mu \]
\[ \Rightarrow f^{-1}(\theta^l) f^{-1}(\sigma^l) \subset \mu \]
\[ \Rightarrow \text{either } f^{-1}(\theta^l) \subset \mu \text{ or } f^{-1}(\sigma^l) \subset \mu \text{ since } \mu \text{ is fuzzy prime ideal} \]
\[ \Rightarrow ff^{-1}(\theta^l) \subset f(\mu) \text{ or } ff^{-1}(\sigma^l) \subset f(\mu) \]
\[ \Rightarrow \theta^l \subset f(\mu) \text{ or } \sigma^l \subset f(\mu) \]
\[ \Rightarrow f(\mu) \text{ is a fuzzy prime ideal of } R^l \]
3.5 Proposition:
Let \( f : R \rightarrow R' \) be a surjective anti-homomorphism. If \( \mu \) is an \( f \)-invariant ideal of \( R \) and \( \mu \) fuzzy primary ideal of \( R \), then \( f(\mu) \) is a fuzzy primary ideal of \( R' \).

Proof:
Let \( \sigma' \) and \( \theta' \) be any two fuzzy ideals of \( R' \) such that \( \theta' \subset \sqrt{f(\mu)} \) with \( \sigma' \not\subset f(\mu) \)
\[
\Rightarrow \theta' \subset \sqrt{f(\mu)}
\]
\[
f^{-1}(\sigma') \subset f^{-1} \sqrt{f(\mu)} = \sqrt{\mu}
\]
This implies that \( f^{-1}(\sigma') \) not subset of \( \mu \) because \( f \) is anti-homomorphism
\[
\Rightarrow f^{-1}(\theta') \subset \sqrt{\mu}
\]
\[
\Rightarrow \theta' \subset f f^{-1}(\theta') \subset f(\sqrt{\mu}) = \sqrt{f(\mu)}
\]
Therefore \( f(\mu) \) is a fuzzy primary ideal of \( R' \).

References


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