Some Remarks on Graded Weak Multiplication Modules\textsuperscript{1}

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Abstract

Let $G$ be a group, $R$ be a $G$-graded ring and $M$ be a $G$-graded $R$-module. In this paper, we study the relation between the category of gr-$R$-modules and their identity components for the weak multiplication property. Also, we introduce some results concerning graded prime submodules.

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Introduction

Let $R$ be a commutative ring with unity 1 and $M$ be a left $R$-module. If $N$ is an $R$-submodule of $M$, then the ideal $\{r \in R : rM \subseteq N\}$ of $R$ will be denoted

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by \((N:_{R} M)\). A commutative ring \(R\) with unity 1 is a \(G\) - graded ring if there exist additive subgroups \(R_g\) of \(R\) indexed by the elements \(g \in G\) such that \(R = \bigoplus_{g \in G} R_g\) and \(R_gR_h \subseteq R_{gh}\) for all \(g, h \in G\). We denote this by \((R, G)\) and we consider \(\text{supp}(R, G) = \{g \in G : R_g \neq 0\}\). The elements of \(R_g\) are called homogeneous of degree \(g\). If \(x \in R\), then \(x\) can be written uniquely as \(\sum_{g \in G} x_g\), where \(x_g\) is the component of \(x\) in \(R_g\). Moreover, \(R_e\) is a subring of \(R\) and \(1 \in R_e\). Also, if \(x \in R_g\) and \(x\) is unit, then \(x^{-1} \in R_{g^{-1}}\). Finally, \(h(R) = \bigcup_{g \in G} R_g\). Let \(M\) be a left \(R\)-module. Then \(M\) is a \(G\) - graded \(R\) - module (in short, \(M\) is \(\text{gr} - R\) - module) if there exist additive subgroups \(M_g\) of \(M\) indexed by the elements \(g \in G\) such that \(M = \bigoplus_{g \in G} M_g\) and \(R_gM_h \subseteq M_{gh}\) for all \(g, h \in G\). We denote this by \((M, G)\) and we consider \(\text{supp}(M, G) = \{g \in G : M_g \neq 0\}\). The elements of \(M_g\) are called homogeneous of degree \(g\). If \(x \in M\), then \(x\) can be written uniquely as \(\sum_{g \in G} x_g\), where \(x_g\) is the component of \(x\) in \(M_g\). Clearly, \(M_g\) is \(R_e\) - submodule of \(M\) for all \(g \in G\). Finally, \(h(M) = \bigcup_{g \in G} M_g\). For more details, one can look in [2, 3].

In Section 1, we give some basic concepts concerning graded modules and graded rings.

In Section 2, we introduce the relations between graded weak multiplication modules and their identity components. Also, we introduce some results concerning graded prime submodules.

1. Preliminaries

In this section, we give some basic concepts concerning graded modules and graded rings.

**Definition 1.1.** [2] \((R, G)\) is called strong if \(R_gR_h = R_{gh}\) for all \(g, h \in G\).

**Definition 1.2.** [2] \((M, G)\) is called strong if \(R_gM_h = M_{gh}\) for all \(g, h \in G\).

**Proposition 1.3.** [2] Let \(R\) be a \(G\) - graded ring. Then \((R, G)\) is strong if and only if every \(\text{gr} - R\) - module is strongly graded.

**Definition 1.4.** [3] \((R, G)\) is called first strong if \(1 \in R_gR_{g^{-1}}\) for all \(g \in \text{supp}(R, G)\).

**Proposition 1.5.** [3] Let \(R\) be a \(G\) - graded ring. Then \((R, G)\) is first strong if and only if \(\text{supp}(R, G)\) is a subgroup of \(G\) and \(R_gR_h = R_{gh}\) for all \(g, h \in \text{supp}(R, G)\).

**Definition 1.6.** [3] \((M, G)\) is called first strong if \(\text{supp}(R, G)\) is a subgroup of \(G\) and \(R_gM_h = M_{gh}\) for all \(g \in \text{supp}(R, G)\), \(h \in G\).
Proposition 1.7. [3] Let $R$ be a $G$-graded ring. Then $(R, G)$ is first strong if and only if every gr-$R$-module is first strongly graded.

Definition 1.8. [2] Let $R$ be a $G$-graded ring and $I$ be an ideal of $R$. Then $I$ is called $G$-graded ideal if $I = \bigoplus_{g \in G}(I \cap R_g)$.

Definition 1.9. [2] Let $M$ be a $G$-gr-$R$-module and $N$ be an $R$-submodule of $M$. Then $N$ is called $G$-gr-$R$-submodule if $N = \bigoplus_{g \in G}(N \cap M_g)$.

Definition 1.10. [1] Let $M$ be a gr-$R$-module and $N \neq M$ be a gr-$R$-submodule of $M$. Then $N$ is called graded prime $R$-submodule of $M$ if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $m \in N$ or $r \in (N :_R M)$.

Definition 1.11. [1] Let $I$ be a proper gr-ideal of a graded ring $R$. Then $I$ is called graded prime ideal of $R$ if whenever $a, b \in h(R)$ with $ab \in I$, then either $a \in I$ or $b \in I$.

Definition 1.12. [4] $(R, G)$ is called crossed product over the support if $R_g$ contains a unit for all $g \in \text{supp}(R, G)$.

Proposition 1.13. [4] If $(R, G)$ is crossed product over the support then $(R, G)$ is first strong.

2. Graded Weak Multiplication Modules

In this section, we introduce the relations between graded weak multiplication modules and their identity components. Also, we introduce some results concerning graded prime submodules. Firstly, we begin with the following well-known result, however, we introduce the proof.

Lemma 2.1. Let $M$ be a gr-$R$-module. If $N$ is a graded prime $R$-submodule of $M$, then $(N :_R M)$ is a graded prime ideal of $R$.

Proof. Firstly, we prove that $(N :_R M)$ is a graded ideal of $R$. Let $x \in (N :_R M)$. Then $x \in R$ and then $x = \sum_{i=1}^{n} x_{gi}$, where $g_i \in R_{g_i} = \{0\}$, $g_i \neq g_j$ for all $i \neq j$, $1 \leq i, j \leq n$. Let $m \in M$. Then $xm \in xM \subseteq N$. So, $x = \sum_{i=1}^{n} x_{gi}m \in N$ and since $N$ is graded, $x_{gi}m \in N$ for all $1 \leq i \leq n$. Hence, $x_{gi}M \subseteq N$ for all $1 \leq i \leq n$ and then $x_{gi} \in (N :_R M)$ for all $1 \leq i \leq n$. Thus, $(N :_R M)$ is graded. Now, let $a, b \in h(R)$ with $ab \in (N :_R M)$. Then $abM \subseteq N$. Since $N$ is graded prime, $N \neq M$ and then there exists
$m \in M - N$. Now, $abm \in abM \subseteq N$ and since $N$ is graded prime, either $bm \in N$ or $a \in (N:_R M)$. If $bm \in N$, then since $N$ is graded prime and $m \notin N$, $b \in (N:_R M)$. Hence $(N:_R M)$ is graded prime. \hfill \square

**Definition 2.2.** Let $M$ be a gr - $R$ - module. Then $M$ is called graded weak multiplication module (shortly, gr - $R$ - weak multiplication module) if for every graded prime $R$ - submodule $N$ of $M$, $N = IM$ for some graded prime ideal $I$ of $R$. Using Lemma 2.1, one can get $I = (N:_R M)$.

**Proposition 2.3.** Let $M$ be a gr - $R$ module. If $M$ is weak multiplication as an $R_e$ - module, then $M$ is gr - $R$ - weak multiplication.

**Proof.** Let $N$ be a graded prime $R$ - submodule of $M$. Then $R_eN = N$ is a prime $R_e$ - submodule of $M$. Since $M$ is $R_e$ - weak multiplication, $N = (N:_R M)M \subseteq (N:_R M)M$. Let $m \in (N:_R M)M$. Then $m = rx$ for some $r \in (N:_R M)$ and $x \in M$. So, $m = rx \in rM \subseteq N$ and then $(N:_R M)M \subseteq N$. Hence, $N = (N:_R M)M$. Thus, $M$ is gr - $R$ - weak multiplication. \hfill \square

**Lemma 2.4.** Let $M$ be a $G$ - gr - $R$ - module. For $g \in G$, if $N$ is a prime $R_e$ - submodule of $M_g$, then $RN$ is a graded prime $R$ - submodule of $M$.

**Proof.** Clearly, $RN$ is a gr - $R$ - submodule of $M$. Let $r \in h(R)$ and $m \in h(M)$ with $rm \in RN$. Then for $g \in G$, $r_gm_g = (rm)_g \in (RN)_g = R_eN = N$. Since $N$ is prime, either $m_g \in N$ or $r_gM_g \subseteq N$. If $m_g \in N$, then since $g$ is arbitrary and $N$ is submodule, $m = \sum_{g \in G} m_g \in N$ and then $m = 1.m \in RN$. If $r_gM_g \subseteq N$, then since $g$ is arbitrary, $(rm)_g = r_gM_g \subseteq N$ for all $g \in G$ and then $rM \subseteq N$, so $rM = 1.rM \subseteq RN$. Hence, $RN$ is a graded prime $R$ - submodule of $M$. \hfill \square

**Proposition 2.5.** Let $M$ be a $G$ - gr - $R$ - weak multiplication module. Then $M_g$ is $R_e$ - weak multiplication module for all $g \in G$.

**Proof.** Let $g \in G$ and $N$ be a prime $R_e$ - submodule of $M_g$. Then by Lemma 2.4, $RN$ is a graded prime $R$ - submodule of $M$. Since $M$ is gr - $R$ - weak multiplication, $RN = (RN:_R M)M$ and then $(RN)_g = ((RN:_R M)M)_g$. But

$$(RN)_g = RN \cap M_g = R_eN = N$$

and

$$(RN:_R M)_g = (RN:_R M)M \cap M_g = (R_eN :_{R_e} M_g)M_g = (N :_{R_e} M_g)M_g.$$  

So, by Lemma 2.1, $(N :_{R_e} M_g)$ is a prime ideal of $R_e$ with $N = (N :_{R_e} M_g)M_g$. Hence, $M_g$ is $R_e$ - weak multiplication. \hfill \square
Lemma 2.6. Let $R$ be a first strongly $G$ - graded ring and $M$ be a gr - $R$ - module. If $N$ is a graded prime - $R$ - submodule of $M$, then $N_g$ is a prime $R_e$ - submodule of $M_g$ for all $g \in \text{supp}(R, G)$.

Proof. Let $g \in \text{supp}(R, G)$, $r_e \in R_e$ and $m_g \in M_g$ such that $r_em_g \in N_g$. Since $(rm)_g = r_em_g$, $r \in h(R)$ and $m \in h(M)$ with $rm \in N$. Since $N$ is graded prime, either $m \in N$ or $r \in (N :_R M)$. If $m \in N$, then $m_g \in N_g$. If $r \in (N :_R M)$, then $r_e \in (N :_R M)_e = (N :_R M) \cap R_e = (N_e :_R e M_e)$ and then $r_e M_g = r_e R_g M_e = R_g r_e M_e \subseteq R_g N_e \subseteq N_g$, i.e., $r_e \in (N_g :_R e M_g)$. Hence $N_g$ is prime.

Proposition 2.7. Let $R$ be a strongly $G$ - graded ring and $M$ be a gr - $R$ - module. If $M_e$ is $R_e$ - weak multiplication, then $M$ is gr - $R$ - weak multiplication.

Proof. Let $N$ be a graded prime $R$ - submodule of $M$. Then by Lemma 2.6, $N_e$ is a prime $R_e$ - submodule of $M_e$. Since $M_e$ is $R_e$ - weak multiplication, $N_e = (N_e :_R e M_e) M_e$. Let $g \in G$. Then $N_g = R_e N_g = R_g R_{g^{-1}} N_g \subseteq R_g N_e \subseteq N_g$. So, $N_g = R_g N_e$ for all $g \in G$. Now, $N_g = R_g N_e = R_g (N_e :_R e M_e) M_e = (N_e :_R e M_e) R_g M_e = (N_e :_R e M_e) M_g$ for all $g \in G$. So, $N = (N_e :_R e M_e) M$ and then $N = RN = R(N_e :_R e M_e) M$ where $R(N_e :_R e M_e)$ is graded prime ideal of $R$ (see Lemma 2.1 and Lemma 2.4). Therefore, $M$ is gr - $R$ - weak multiplication.

Lemma 2.8. Suppose $(R, G)$ is a crossed product over the support and $M$ is a gr - $R$ - module. For $g \in \text{supp}(R, G)$, if $N$ is a prime $R_e$ - submodule of $M_g$, then $R_{g^{-1}} N$ is a prime $R_{e^{-1}}$ - submodule of $M_e$.

Proof. Clearly, $R_{g^{-1}} N$ is an $R_{e^{-1}}$ - submodule of $M_e$. Let $r \in R_e$ and $m \in M_e$ with $rm \in R_{g^{-1}} N$. Since $(R, G)$ is crossed product over the support, $R_g$ contains a unit, say $x$ and by Proposition 1.13, $(R, G)$ is first strong. Now $r(xm) = x(rm) \in R_g R_{g^{-1}} N = R_e N = N$. Since $N$ is prime, either $xm \in N$ or $rM_g \subseteq N$. If $xm \in N$, then $m = 1.m = x^{-1}(xm) \in R_{g^{-1}} N$. If $rM_g \subseteq N$, then $rM_e = rR_{g^{-1}} M_g = R_{g^{-1}} rM_g \subseteq R_{g^{-1}} N$. Hence, $R_{g^{-1}} N$ is a prime $R_{e^{-1}}$ - submodule of $M_e$.

Proposition 2.9. Suppose $(R, G)$ is a crossed product over the support and $M$ is a gr - $R$ - module. Then $M_e$ is $R_e$ - weak multiplication if and only if $M_g$ is $R_e$ - weak multiplication for all $g \in \text{supp}(R, G)$.

Proof. Suppose $M_e$ is $R_e$ - weak multiplication. Let $g \in \text{supp} (R, G)$ and let $N$ be a prime $R_e$ - submodule of $M_g$. Then by Lemma 2.8, $R_{g^{-1}} N$ is a prime $R_{e^{-1}}$ -
submodule of $M_e$. Since $M_e$ is $R_e$-weak multiplication, $R_{g^{-1}}N = IM_e$ for some prime ideal $I$ of $R_e$ and then $N = R_e N = R_{g} R_{g^{-1}} N = R_{g} IM_e = IR_{g} M_e = IM_g$. Hence $M_g$ is $R_e$-weak multiplication. The converse is obvious.

References


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