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Integrals of Fractional Parts and Some New Identities on Bernoulli Numbers

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Abstract

In this paper, we calculate the values of the integrals $\int_0^1 \{\frac{1}{x}\}^m dx$, $\int\int_{0\le x,y\le 1} \{\frac{1}{x+y}\}^m dxdy$, $\int\int\int_{0\le x,y,z\le 1} \{\frac{1}{x+y+z}\}^m dxdydz$ and $\int_0^1 \{\frac{1}{x}\}^m \{\frac{1}{1-x}\}^n dx$, where m and n are positive integers and $\{u\}$ is the fractional part of u, and express their values in terms of Euler's constant and Riemann-Zeta function. We also obtain a set of identities involving the Bernoulli and Harmonic numbers.

Keywords: fractional parts, new identities, Euler's constant, Bernoulli numbers, Harmonic numbers

1 Introduction

In [2], Havil studied the integral of $\int_0^1 \{\frac{1}{x}\}dx$ and expressed its value in terms of Euler's constant. In [3], Qin calculated the value of $\int_0^1 \int_0^1 \{\frac{1}{x+y}\}dxdy$. Recently, Furdui [5] posted a problem for calculating the values of the integrals

$\int_0^1 \{\frac{1}{x}\}^3$ and $\int_0^1 \{\frac{1}{x}\}^4 dx$ and some results have been published. In this paper, we generalize Furdui's question and calculate the values of the integrals

$$\int_0^1 \{\frac{1}{x}\}^m dx, \tag{1.1}$$

$$\int\int_{0\le x,y\le 1} \{\frac{1}{x+y}\}^m dxdy \tag{1.2}$$

$$\int\int\int_{0\le x,y,z\le 1} \{\frac{1}{x+y+z}\}^m dxdydz, \tag{1.3}$$

and

$$\int_0^1 \{\frac{1}{x}\}^m \{\frac{1}{1-x}\} dx, \tag{1.4}$$

where $\{u\}$ represents the fractional part of u, for all positive integers m and n.During the process of our calculation, we have also obtained infinitely many identities involving Bernoulli numbers and Harmonic numbers. With our limited knowledge, we believe that some of these identities are new.

2 Calculation of Integral (1.1)

Theorem 1

$$\int_0^1 \{\frac{1}{x}\}^m dx = \begin{cases} 1-\gamma, \ \textit{if } m=1, \\ \ln(2\pi) - \frac{m}{2}\gamma - \frac{1}{m-1} - \sum_{k=1}^{[\frac{m-2}{2}]} \frac{(-1)^k (m)_{2k+1} \zeta(2k+1)}{2^{2k+1}\pi^{2k}} \\ +2\sum_{k=1}^{[\frac{m-1}{2}]} \frac{(-1)^{k-1}(m)_{2k}\zeta'(2k)}{(2\pi)^{2k}}, \ \textit{for any integer } m>1. \end{cases}$$

where $(m)_k = \frac{m!}{(m-k)!}$ *and* $\zeta(x)$ *is the Riemann-Zeta function.*

When $m = 3$ and 4,this theorem answers Furdui's question [5]. In fact, we have

$$\begin{aligned}\int_0^1 \left\{\frac{1}{x}\right\}^2 dx &= \ln(2\pi) - \gamma - 1,\\ \int_0^1 \left\{\frac{1}{x}\right\}^3 dx &= \ln(2\pi) - \frac{3}{2}\gamma - \frac{1}{2} + \frac{3\zeta'(2)}{\pi^2},\\ \int_0^1 \left\{\frac{1}{x}\right\}^4 dx &= \ln(2\pi) - 2\gamma - \frac{1}{3} + \frac{3\zeta(3)}{\pi^2} + \frac{6\zeta'(2)}{\pi^2}.\end{aligned}$$

These results improve the answer given by Janous[5] which are in terms of infinite series of Riemann-Zeta functions. We first prove a sequence of lemmas.

Lemma 2 *For any integer $m > 0$, $\int_0^1 \{\frac{1}{x}\}^m dx = I_m$ where*

$$I_m = \sum_{n=1}^{\infty}(m\sum_{k=1}^{m-1}\frac{(-n)^{k-1}}{m-k} + (-1)^{m-1}mn^{m-1}\ln\frac{n+1}{n} - \frac{1}{n+1}). \tag{2.1}$$

Proof.

$$\begin{aligned}\int_0^1 \left\{\frac{1}{x}\right\}^m dx &= \int_1^{\infty}\frac{\{u\}^m}{u^2}du = \sum_{n=1}^{\infty}\int_n^{n+1}\frac{(u-n)^m}{u^2}du\\ &= \sum_{n=1}^{\infty}\int_0^1 \frac{u^m}{(u+n)^2}du\\ &= \sum_{n=1}^{\infty}\left(m\int_0^1\frac{u^{m-1}du}{u+n} - \frac{1}{n+1}\right)\end{aligned}$$

$$\begin{aligned}&= \sum_{n=1}^{\infty}\left[m\int_0^1\left(\sum_{k=1}^{m-1}(-n)^{k-1}u^{m-k-1} + \frac{(-n)^{m-1}}{u+n}\right)du - \frac{1}{n+1}\right]\\ &= \sum_{n=1}^{\infty}\left(m\sum_{k=1}^{m-1}\frac{(-n)^{k-1}}{m-k} + (-n)^{m-1}m\ln\frac{n+1}{n} - \frac{1}{n+1}\right).\end{aligned}$$

■

To prepare for the proof of the other lemmas, we list several formulas related to zeta function $\zeta(x)$, gamma function $\Gamma(x)$, Bernoulli numbers B_k and harmonic numbers H_k [1].

$$\zeta(2m) = (-1)^{m-1}\frac{(2\pi)^{2m}B_{2m}}{2(2m)!} \text{ for integer } m > 0. \tag{2.2}$$

$$\int_0^\infty \frac{y^{\alpha-1}}{e^{2\pi y}-1}dy = \frac{\Gamma(\alpha)\zeta(\alpha)}{(2\pi)^\alpha}, \text{ for } \operatorname{Re}\alpha > 1. \tag{2.3}$$

$$\int_0^\infty \frac{y^{\alpha-1}\ln y}{e^{2\pi y}-1}dy = \frac{\Gamma'(\alpha)\zeta(\alpha)+\Gamma(\alpha)\zeta'(\alpha)-\ln(2\pi)\Gamma(\alpha)\zeta(\alpha)}{(2\pi)^\alpha}, \text{ for } \operatorname{Re}\alpha > 1. \tag{2.4}$$

$$\frac{\Gamma'(m)}{\Gamma(m)} = H_{m-1}-\gamma,\ H_m = \sum_{k=1}^m \frac{1}{k} \text{ for integer } m > 0. \tag{2.5}$$

Combining (2.1)-(2.4), we can get the following formulas.

$$\int_0^\infty \frac{y^{2m-1}}{e^{2\pi y}-1}dy = \frac{(-1)^{m-1}B_{2m}}{4m}, \text{ for integer } m > 0. \tag{2.6}$$

$$\begin{aligned}\int_0^\infty \frac{y^{\alpha-1}\ln y}{e^{2\pi y}-1}dy &= (-1)^{m-1}B_{2m}\frac{H_{2m-1}-\gamma-\ln(2\pi)}{4m} + \frac{(2m-1)!\zeta'(2m)}{(2\pi)^{2m}},\\ \text{for integer } m &> 0.\end{aligned} \tag{2.7}$$

The Abel-Plana formula [1], regular Taylor expansion and formula (2.6) can be used to prove the following lemma..

Lemma 3 *Let a and n be two non-negative integers with $a < n$ and $S = \{z|a \le \operatorname{Re} z \le n\}$. Assume that $f(z)$ is a function defined in S satisfying the following conditions.*

1). $f(t)$ is real over $a \le t \le n$;

2). $f(z)$ is continuous throughout S and holomorphic in the interior of S;

3). $f(z) = o(e^{2\pi|\operatorname{Im} z|})$ as $\operatorname{Im} z \to \pm\infty$ in S uniformly with respect to $\operatorname{Re} z$.

Then,

$$\begin{aligned}\sum_{k=a}^n f(k) &= \int_a^n f(x)dx + \frac{f(n)+f(a)}{2} - 2\int_0^\infty \frac{\operatorname{Im} f(a+iy)}{e^{2\pi y}-1}dy \\ &+\sum_{j=1}^M \frac{B_{2j}}{(2j)!}f^{(2j-1)}(n) + \frac{2(-1)^M}{(2M)!}\int_0^\infty \frac{y^{2M}\operatorname{Im} f^{(2M)}(n+i\theta_n y)}{e^{2\pi y}-1}dy,\end{aligned} \tag{2.8}$$

where M is a positive integer and $0 < \theta_n < 1$.

By using the Leibniz rule for differentiation, one may obtain the following lemma.

Lemma 4 *For any integers k and m with $m \geq k \geq 0$,*

$$\frac{d^k}{dx^k}(x^m \ln x) = (m)_k x^{m-k} \ln x + C_{m,k} x^{m-k}, \tag{2.10}$$

where $C_{m,k} = k! \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \begin{pmatrix} m \\ k-j \end{pmatrix}$.

Using this lemma and mathematical induction, one can also get next lemma.

Lemma 5 *For any integers m and k with $m \geq k \geq 0$,*

$$\frac{d^k}{dx^k}(x^m \ln x) = (m)_k x^{m-k}(\ln x + \sum_{j=0}^{k-1} \frac{1}{m-j}).$$

In particular, $C_{m,m} = \begin{cases} 0, \text{ if } m = 0, \\ m!H_m, \quad \text{if } m > 0, \end{cases}$ *and* $C_{m,m-1} = m!(H_m - 1)$.

Now, we combine the lemmas above to obtain a few summation formulas.

Lemma 6 *For any integers $m > 0$, $N > 0$ and $i = 0$ or1,the following formulas are true.*

$$\begin{aligned} \sum_{k=1}^{N} k^{2m-i} &= \frac{N^{2m+1-i}}{2m+1-i} + \frac{N^{2m-i}}{2} \\ &\quad + \frac{1}{2m+1-i} \sum_{j=1}^{m-i} C_{2m+1-i}^{2j-1+i} B_{2m-2j+2-2i} N^{2j-1+i}; \end{aligned} \tag{2.11}$$

$$\begin{aligned} \sum_{k=1}^{N} k^{2m-1} \ln k &= \frac{N^{2m} \ln N}{2m} + \frac{N^{2m-1} \ln N}{2} + \frac{1}{2m} \sum_{j=1}^{m} C_{2m}^{2j-2} B_{2m-2j+2} N^{2j-2} \ln N \\ &\quad - \frac{N^{2m}}{4m^2} + \sum_{j=1}^{m-1} \frac{C_{2m-1,2m-2j-1} B_{2m-2j}}{(2m-2j)!} N^{2j} + \frac{\gamma + \ln(2\pi)}{2m} B_{2m} \\ &\quad + 2(-1)^m \frac{(2m-1)!\zeta'(2m)}{(2\pi)^{2m}} + O(\frac{1}{N}); \end{aligned} \tag{2.12}$$

$$\sum_{k=1}^{N} k^{2m}\ln k = \frac{N^{2m+1}\ln N}{2m+1}+\frac{N^{2m}\ln N}{2}+\frac{1}{2m+1}\sum_{j=1}^{m} C_{2m+1}^{2j-1}B_{2m-2j+2}N^{2j-1}\ln N$$
$$-\frac{N^{2m+1}}{(2m+1)^2}+\sum_{j=1}^{m}\frac{C_{2m,2m-2j+1}B_{2m-2j+2}}{(2m-2j+2)!}N^{2j-1}$$
$$+(-1)^{m-1}\frac{(2m)!\zeta(2m+1)}{2^{2m+1}\pi^{2m}}+O(\frac{1}{N}). \qquad (2.13)$$

Proof. Let $f(x)=x^{2m-1}$ and $M=m$. Then, (2.8) becomes

$$\sum_{k=0}^{N} k^{2m-1} = \int_0^N x^{2m-1}dx+\frac{N^{2m-1}}{2}+2(-1)^m\int_0^\infty \frac{y^{2m-1}}{e^{2\pi y}-1}dy$$
$$+\sum_{j=1}^{m}\frac{B_{2j}(2m-1)_{2j-1}N^{2m-2j}}{(2j)!}$$
$$= \frac{N^{2m}}{2m}+\frac{N^{2m-1}}{2}-\frac{B_{2m}}{2m}+\frac{1}{2m}\sum_{j=1}^{m}\begin{pmatrix}2m\\2j\end{pmatrix}B_{2m-2j}N^{2j}$$
$$= \frac{N^{2m}}{2m}+\frac{N^{2m-1}}{2}+\frac{1}{2m}\sum_{j=1}^{m-1}\begin{pmatrix}2m\\2j\end{pmatrix}B_{2m-2j}N^{2j}.$$

If we let $f(x)=x^{2m}$ and $M=m+1$ in (2.8), formula (2.11) for $i=0$ can be obtained similarly. To prove (2.12), we let $f(x)=\begin{cases}x^{2m-1}\ln x, & \text{for } 0<x\le N,\\ 0, & \text{for } x=0.\end{cases}$ Then, $f(x)$ satisfies all conditions in Lemma 3 with $a=0$ and $n=N$ and, with $M=m$, (2.8) can be written as

$$\sum_{k=1}^{N} k^{2m-1}\ln k = \sum_{k=0}^{N} f(k) \qquad (2.14)$$
$$= \int_0^N x^{2m-1}\ln x dx+\frac{N^{2m-1}\ln N}{2}+2(-1)^m\int_0^\infty\frac{y^{2m-1}\ln y}{e^{2\pi y}-1}dy$$
$$+\sum_{j=1}^{m}\frac{B_{2j}}{(2j)!}(x^{2m-1}\ln x)^{(2j-1)}|_{x=N}+O(\frac{1}{N}).$$

Applying (2.7) and (2.10) to this expression, one can get formula (2.12). The proof of (2.13) is similar. ■

Now, we are ready to complete the proof of Theorem 1. By using (2.1), we first have

$$I_1=\sum_{n=1}^{\infty}(\ln\frac{n+1}{n}-\frac{1}{n+1})=\lim_{n\to\infty}(\ln n-\sum_{k=1}^{n}\frac{1}{k}+1)=1-\gamma.$$

For $m>1$, one may apply the Binomial Theorem to $n^{m-1}=(n+1-1)^{m-1}$ and rewrite (2.1) into

$$I_m = \lim_{N\to\infty}\sum_{n=1}^{N-1}(m\sum_{k=1}^{m-1}\frac{(-n)^{k-1}}{m-k}+m\sum_{k=1}^{m-2}(-1)^k\begin{pmatrix} m-1\\ k\end{pmatrix}(n+1)^k\ln(n+1)$$
$$+m\ln(n+1)+(-1)^{m-1}m(n+1)^{m-1}\ln(n+1)+(-1)^m mn^{m-1}\ln n-\frac{1}{n+1})$$

$$= \lim_{N\to\infty}\left(\begin{array}{c} \frac{m(N-1)}{m-1}+m\sum\limits_{k=1}^{\left[\frac{m-1}{2}\right]}\frac{N^{2k-1}}{m-2k}-m\sum\limits_{k=1}^{\left[\frac{m-2}{2}\right]}\frac{N^{2k}}{m-2k-1}\\ -m\sum\limits_{k=1}^{\left[\frac{m-1}{2}\right]}\frac{1}{m-2k}\sum\limits_{n=1}^{N}n^{2k-1}-m\sum\limits_{k=1}^{\left[\frac{m-1}{2}\right]}\begin{pmatrix} m-1\\ 2k-1\end{pmatrix}\sum\limits_{n=1}^{N}n^{2k-1}\ln n\\ +m\sum\limits_{k=1}^{\left[\frac{m-2}{2}\right]}\frac{1}{m-2k-1}\sum\limits_{n=1}^{N}n^{2k}+m\sum\limits_{k=1}^{\left[\frac{m-2}{2}\right]}\begin{pmatrix} m-1\\ 2k\end{pmatrix}\sum\limits_{n=1}^{N}n^{2k}\ln n\\ +m\ln N!+(-1)^{m-1}mN^{m-1}\ln N-\sum\limits_{n=1}^{N}\frac{1}{n}+1 \end{array}\right). \tag{2.15}$$

Applying Lemma 6, Stirling's formula [4] and the formula $\sum\limits_{n=1}^{N}\frac{1}{n}=\ln N+\gamma+O(\frac{1}{N})$ to (2.15), we get

$$I_m = \lim_{N\to\infty}(\sum_{k=1}^{m-1}a_kN^k+\sum_{k=0}^{m-1}b_kN^k\ln N-(\gamma+\ln(2\pi))\sum_{k=1}^{\left[\frac{m-1}{2}\right]}\begin{pmatrix} m\\ 2k\end{pmatrix}B_{2k}$$
$$-m\sum_{k=1}^{\left[\frac{m-1}{2}\right]}\begin{pmatrix} m-1\\ 2k-1\end{pmatrix}2(-1)^k\frac{(2k-1)!\zeta'(2k)}{(2\pi)^{2k}}$$
$$+m\sum_{k=1}^{\left[\frac{m-2}{2}\right]}\begin{pmatrix} m-1\\ 2k\end{pmatrix}(-1)^{k-1}\frac{(2k)!\zeta(2k+1)}{2^{2k+1}\pi^{2k}}+\frac{m}{2}\ln(2\pi)-\gamma-\frac{1}{m-1}+O(\frac{1}{N}),$$

where

$$a_k = (-1)^{k-1}(\frac{m}{2(m-k-1)}+\frac{m}{k(m-k)}+\sum_{j=1}^{\left[\frac{m-k-1}{2}\right]}(m\begin{pmatrix} m-1\\ k+2j-1\end{pmatrix}\frac{C_{k+2j-1,2j-1}}{(2j)!}$$
$$-\frac{1}{k}\begin{pmatrix} m\\ k\end{pmatrix}+\frac{m}{(m-k-2j)(k+2j)}\begin{pmatrix} k+2j\\ k\end{pmatrix})B_{2j}),$$

for $k = 1, 2, 3, ..., m-1$ and

$$b_k = (-1)^{k-1} \begin{pmatrix} m \\ k \end{pmatrix} \left(\sum_{j=1}^{\left[\frac{m-k-1}{2}\right]} \begin{pmatrix} m-k \\ 2j \end{pmatrix} B_{2j} + 1 - \frac{m-k}{2} \right) \text{ for } k = 0, 1, 2..., m-1.$$

Since I_m is bounded and all a_k and b_k are independent of N, we have $a_k = 0$ for all $k = 1, 2, ..., m-1$ and $b_k = 0$ for all $k = 0, 1, 2, ..., m-1$. Therefore,

$$\begin{aligned} I_m &= -(\gamma + \ln(2\pi)) \sum_{j=1}^{\left[\frac{m-1}{2}\right]} \begin{pmatrix} m \\ 2k \end{pmatrix} B_{2k} - 2m \sum_{j=1}^{\left[\frac{m-1}{2}\right]} \begin{pmatrix} m-1 \\ 2k-1 \end{pmatrix} (-1)^k \frac{(2k-1)!\zeta'(2k)}{(2\pi)^{2k}} \\ &+ m \sum_{j=1}^{\left[\frac{m-2}{2}\right]} \begin{pmatrix} m-1 \\ 2k \end{pmatrix} (-1)^{k-1} \frac{(2k)!\zeta(2k+1)}{2^{2k+1}\pi^{2k}} + \frac{m}{2}\ln(2\pi) - \gamma - \frac{1}{m-1}. \end{aligned}$$

Using the property $b_1 = 0$, we complete the proof.

Equations $a_k = 0$ for $k = 1, 2, ..., m-1$ and $b_k = 0$ for $k = 0, 1, 2, ..., m-1$ give us infinitely many identities involving the Bernoulli and Harmonic numbers. In particular, we get the following results.

Theorem 7 *For any integer $m \geq 2$, the following equations are true.*

$$\sum_{k=1}^{\left[\frac{m-1}{2}\right]} \begin{pmatrix} m \\ 2k \end{pmatrix} B_{2k} = \frac{m}{2} - 1, \tag{2.16}$$

$$\sum_{k=1}^{\left[\frac{m-1}{2}\right]} \left(\frac{1}{m-2k} + H_{2k} \begin{pmatrix} m \\ 2k \end{pmatrix}\right) B_{2k} = \frac{m}{2} - \frac{1}{m} - \frac{1}{2(m-1)}. \tag{2.17}$$

Identity (2.16) is a direct result of $b_0 = 0$. Identity (2.17) can be obtained by replacing m by $m-1$ and using (2.16) in $a_1 = 0$.

3 Calculation of (1.2), (1.3) and (1.4)

Theorem 8 *For any integer $m > 0$, we have*

$$\int\int_{0 \leq x, y \leq 1} \left\{ \frac{1}{x+y} \right\}^m dxdy = \begin{cases} 2\ln 2 - \frac{\pi^2}{12}, & \text{if } m = 1, \\ \frac{3}{2} - \frac{\pi^2}{12} - \ln 2 - \gamma, & \text{if } m = 2, \\ \frac{1}{2} - \frac{\pi^2}{12} + \frac{m-3+2^{2-m}}{(m-1)(m-2)} + \frac{m}{2} I_{m-1}, & \text{if } m > 2; \end{cases} \tag{3.1}$$

and

$$\int\int\int_{0\le x,y,z\le 1}\left\{\frac{1}{x+y+z}\right\}^m dxdydz=\begin{cases}\frac{9}{2}\ln 3-\frac{13}{24}-\frac{19}{4}\ln 2-\frac{\zeta(3)}{3},\ m=1,\\ \frac{53}{24}+4\ln 2-3\ln 3-\frac{\zeta(3)}{3}-\frac{\pi^2}{12},\ m=2,\\ \frac{269}{96}+\frac{1}{2}\ln 3-\frac{3}{4}\ln 2-\frac{\zeta(3)}{3}-\frac{\pi^2}{6}-\frac{3\gamma}{2}, m=3,\\ \frac{h(m)}{(m-1)(m-2)(m-3)}+\frac{6m+8-8\zeta(3)-m\pi^2}{24}\\ +\frac{m(m-1)}{4}I_{m-2}, m>3.\end{cases} \tag{3.2}$$

where $h(m)=2^{-1}(m^2-4m+2)-2^{-m-1}(3m^2-15m+10)+2^{-m}(m^2-9m+26)-3^{-m}27$.

Proof. For the double integral, we first integrate it along the line L : $x+y=\sqrt{2}t$, and then the line passing through the origin and perpendicular to L. Since $0\le x,y\le 1$ is a square, we need to break the integral into two parts: one for t changing from 0 to $\frac{\sqrt{2}}{2}$ and the other for t changing from $\frac{\sqrt{2}}{2}$ to $\sqrt{2}$. For the first part, we have

$$\begin{aligned}\int\int_{0\le x,y\le 1,x+y\le 1}\left\{\frac{1}{x+y}\right\}^m dxdy &= \int_0^{\frac{\sqrt{2}}{2}}\int_{x+y=\sqrt{2}t}\left\{\frac{1}{x+y}\right\}^m dsdt\\ &= 2\int_0^{\frac{\sqrt{2}}{2}}\left\{\frac{1}{\sqrt{2}t}\right\}^m tdt=\int_0^1 u\left\{\frac{1}{u}\right\}du\\ &= \int_1^\infty \frac{\{u\}^m}{u^3}du=\sum_{n=1}^\infty\int_n^{n+1}\frac{(u-n)^m}{u^3}du\end{aligned}$$

$$\begin{aligned}&= \sum_{n=1}^\infty\int_0^1\frac{u^m}{(u+n)^3}du\\ &= -\frac{1}{2}\sum_{n=1}^\infty\frac{1}{(n+1)^2}+\frac{m}{2}\sum_{n=1}^\infty\int_0^1\frac{u^{m-1}}{(u+n)^2}du\\ &= \frac{1}{2}-\frac{\pi^2}{12}+\frac{m}{2}I_{m-1}.\end{aligned}$$

For $0 \le x, y \le 1$ and $x + y \ge 1$, $0 < \frac{1}{x+y} < 1$and $\left\{\frac{1}{x+y}\right\} = \frac{1}{x+y}$. Therefore, we have

$$\begin{aligned}
\iint\limits_{0\le x,y\le 1, x+y\ge 1} \left\{\frac{1}{x+y}\right\}^m dxdy &= \iint\limits_{0\le x,y\le 1, x+y\ge 1} (\frac{1}{x+y})^m dxdy \\
&= \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \int\limits_{x+y=\sqrt{2}t} \frac{dsdt}{(x+y)^m} \\
&= \sqrt{2}\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \frac{2-\sqrt{2}t}{(\sqrt{2}t)^m} dt = \int_1^2 \frac{2-u}{u^m} du \\
&= \begin{cases} 2\ln 2 - 1, m = 1 \\ 1 - \ln 2, m = 2 \\ \frac{m-3+2^{2-m}}{(m-1)(m-2)}, m > 2. \end{cases}
\end{aligned}$$

Combining the results, we get (3.1). The calculation of (3.2) is similar. The only difference is to perform the integration over the plane $P : x+y+z = \sqrt{3}t$ and then along the line passing through the origin and perpendicular to plane P. The interval for t needs to be broken into three parts: one for t changing from 0 to $\frac{\sqrt{3}}{3}$, the second for t changing from $\frac{\sqrt{3}}{3}$ to $\frac{2\sqrt{3}}{3}$ and the other for t changing from $\frac{2\sqrt{3}}{3}$ to $\sqrt{3}$. ■

Theorem 9 *For any integers $m > 0$ and $n > 0$, we have*

$$\int_0^1 \left\{\frac{1}{x}\right\}^m \left\{\frac{1}{1-x}\right\}^n dx = I(m,n) + I(n,m), \tag{3.3}$$

where

$$\begin{aligned}
I(n,m) = &\sum_{j=1}^{m} \sum_{i=0, i\neq m-j}^{n} \frac{(-1)^{n-i+j-1}}{i+j-m} C_n^i \sum_{l=1}^{n-i} (-1)^l C_{n-i}^l R(n-m+j-l) \\
&+ \sum_{j=1, 0\le m-j\le n}^{m-2} jC_n^{m-j} \sum_{k=0}^{n-m+j-1} (-1)^{j-k-1} C_{n-m+j}^{n-m+j-k} J(k) \\
&+ (-1)^{n+m-1} mn \sum_{l=0}^{n-2} (-1)^{n-l-1} C_{n-1}^l J(l) + (-1)^{m-1}\gamma \\
&- m\sum_{i=1}^{n} \frac{(-1)^{m+n-i}}{i} C_n^i (2^i - 1) - \sum_{i=2}^{n} \frac{(-1)^{m+n-i}}{i-1} C_n^i (2^{i-1} - 1)
\end{aligned}$$

$$+(-1)^m H(n-1) + \frac{m(-1)^{m-1}}{n} + \frac{(-1)^{m-1}n}{n-1} - \frac{(-1)^{m+n}}{2} + (n-m)(-1)^{m+n}\ln 2,$$

$$H(k) = \begin{cases} 1, k > 0 \\ 0, k \le 0 \end{cases}, \quad R(n) = \begin{cases} 0, n \ge 0 \\ \gamma, n = -1 \\ \zeta(-n), n < -1 \end{cases},$$

and

$$J(n) = \begin{cases} \frac{\gamma \ln(2\pi)}{2n} B_n + 2(-1)^{\frac{n+1}{2}} \frac{n!\zeta'(n)}{(2\pi)^n}, \text{ if } n > 0 \text{ is odd} \\ (-1)^{\frac{n}{2}-1} \frac{n!\zeta(n+1)}{2^{n+1}\pi^n}, \text{ if } n > 0 \text{ is even} \\ \frac{\ln(2\pi)}{2}, \text{ if } n = 0. \end{cases}$$

In particular,

$$I(n,n) = \begin{cases} -\ln(2\pi) + \gamma + \frac{3}{2}, \text{ if } n = 1 \\ \sum\limits_{k=1}^{n-1} (-1)^k R(-k) \sum\limits_{j=1}^{n-k} C_n^{k+j} \sum\limits_{i=0}^{n-j-k} \frac{(-1)^{n+i-1} j}{i+j-n} C_{n-k-j}^i \\ -\sum\limits_{l=0}^{n-3} C_n^l J(l) \sum\limits_{j=0}^{n-l-2} (-1)^j (j+l) C_{n-1}^j + n^2 \sum\limits_{l=0}^{n-2} (-1)^{n-l} C_{n-1}^l J(l) \\ +(-1)^{n-1}\gamma - n\sum\limits_{i=0}^{n} \frac{(-1)^i}{i} C_n^i (2^i - 1) - \sum\limits_{l=2}^{n} \frac{(-1)^i}{i} C_n^i (2^{i-1} - 1) \\ +(-1)^n H(n-1) + (-1)^{n-1} + \frac{(-1)^{n-1} n}{n-1} - \frac{1}{2}, \text{ if } n > 1. \end{cases}$$

Proof.

$$\int_0^1 \left\{\frac{1}{x}\right\}^m \left\{\frac{1}{1-x}\right\}^n dx = \int_1^\infty \frac{u\}^m}{u^2} \left\{\frac{u}{u-1}\right\}^n du = \sum_{k=1}^\infty \int_k^{k+1} \frac{(u-k)^m}{u^2} \left\{\frac{1}{u-1}\right\}^n du$$

$$= \sum_{k=1}^\infty \int_0^1 \frac{u^m du}{(u+k)^n (u+k+1)^2} + \int_0^1 \frac{u^m}{(u+1)^2} \left\{\frac{1}{u}\right\}^n du.$$

If we denote $I(m,n) = \sum\limits_{k=1}^\infty \int_0^1 \frac{u^m du}{(u+k)^n (u+k+1)^2}$, then,

$$\int_0^1 \frac{u^m}{(u+1)^2} \left\{\frac{1}{u}\right\}^n du = \int_1^\infty \frac{\{v\}^n dv}{v^m (v+1)^2} = \sum_{k=1}^\infty \int_k^{k+1} \frac{(v-k)^n dv}{v^m (v+1)^2}$$

$$= \sum_{k=1}^\infty \int_0^1 \frac{v^n dv}{(v+k)^m (v+k+1)^2} = I(n,m).$$

Thus, we need to calculate for $I(n,m)$. Using partial fractions and the Binomial Theorem over $u^n=(u+k-k)^n$ and $u^n=(u+k+1-k-1)^n$, we first have

$$\begin{aligned}\frac{u^n}{(u+k)^m(u+k+1)^2} &= u^n\left(\sum_{j=1}^{m}(-1)^{j-1}j(u+k)^{j-m-1}+\frac{(-1)^m m}{u+k+1}+\frac{(-1)^m}{(u+k+1)^2}\right)\\ &= \sum_{j=1}^{m}\sum_{i=0}^{n}(-1)^{n-i+j-1}jk^{n-i}C_n^i(u+k)^{i+j-m-1}\\ &\quad +m\sum_{i=0}^{n}(-1)^{m+n-i}(k+1)^{n-i}C_n^i(u+k+1)^{i-1}\\ &\quad +\sum_{i=0}^{n}(-1)^{m+n-i}(k+1)^{n-i}C_n^i(u+k+1)^{i-2}.\end{aligned}$$

Plugging it back to the definition of $I(n,m)$, one has

$$\begin{aligned}I(n,m) &= \sum_{k=2}^{\infty}\int_0^1(\sum_{j=1}^{m}\sum_{i=0}^{n}(-1)^{n-i+j-1}j(k-1)^{n-i}C_n^i(u+k-1)^{i+j-m-1}\\ &\quad +m\sum_{i=0}^{n}(-1)^{m+n-i}k^{n-i}C_n^i(u+k)^{i-1}\\ &\quad +\sum_{i=0}^{n}(-1)^{m+n-i}k^{n-i}C_n^i(u+k)^{i-2})du\\ &= (-1)^{n+m}\lim_{N\to\infty}\sum_{k=1}^{N-1}(\sum_{i=0}^{n}\sum_{j=1,j\neq m-i}^{m}C_n^i\frac{j(-1)^{m+i+j-1}k^{n-i}(k+1)^{i+j-m}}{i+j-m}\\ &\quad -\sum_{i=0}^{n}\sum_{j=1,j\neq m-i}^{m}C_n^i\frac{j(-1)^{m+i+j-1}k^{n+j-m}}{i+j-m}+\sum_{i=0}^{\min\{m-1,n\}}C_n^i(m-i)k^{n-i}\ln k\\ &\quad -\sum_{i=0}^{\min\{m-1,n\}}C_n^i\,(m-i)k^{n-i}\ln(k+1)+m\sum_{i=1}^{n}C_n^i\frac{(-1)^i}{i}(k+1)^{n-i}(k+2)^i\\ &\quad -m\sum_{i=1}^{n}C_n^i\frac{(-1)^i}{i}(k+1)^n+\sum_{i=2}^{n}C_n^i\frac{(-1)^i}{i-1}(k+1)^{n-i}(k+2)^{i-1}\\ &\quad -\sum_{i=2}^{n}\frac{(-1)^iC_n^i}{i-1}(k+1)^{n-1}+(k+1)^{n-1}[\frac{1}{k+2}+(mk+m-n)\ln\frac{k+2}{k+1}]).\end{aligned}$$

Using the Binomial Theorem, we can rewrite it into

$$
\begin{aligned}
I(n,m) \;=\;& (-1)^{n+m}\lim_{N\to\infty}\sum_{k=1}^{N-1}\Big(\sum_{i=0}^{n}\sum_{j=1,j\neq m-i}^{m}\sum_{l=0}^{n-i}C_n^iC_{n-i}^l\frac{j(-1)^{m-l+i+j-1}(k+1)^{n-l+j-m}}{i+j-m}\\
&-\sum_{i=0}^{n}\sum_{j=1,j\neq m-i}^{m}C_n^i\frac{j(-1)^{m+i+j-1}k^{n+j-m}}{i+j-m}+\sum_{i=0}^{\min\{m-1,n\}}C_n^i(m-i)k^{n-i}\ln k\\
&-\sum_{i=0}^{\min\{m-1,n\}}C_n^i(m-i)\sum_{l=0}^{n-i}(-1)^lC_{n-i}^l(k+1)^{n-i-l}\ln(k+1)\\
&+m\sum_{i=1}^{n}C_n^i\frac{(-1)^i}{i}\sum_{l=0}^{i}(k+1)^{n-i+l}-m\sum_{i=1}^{n}C_n^i\frac{(-1)^i}{i}(k+1)^n\\
&+\sum_{i=2}^{n}C_n^i\frac{(-1)^i}{i-1}\sum_{l=0}^{i-1}C_{i-1}^l(k+1)^{n-i+l}-\sum_{i=2}^{n}C_n^i\frac{(-1)^i}{i-1}(k+1)^{n-1}\\
&+\sum_{l=1}^{n-1}(-1)^lC_{n-1}^l(k+2)^{n-2-l}+m\sum_{l=0}^{n}(-1)^lC_n^l(k+2)^{n-l}\ln(k+2)\\
&-n\sum_{l=0}^{n-1}(-1)^lC_{n-1}^l(k+2)^{n-l-1}\ln(k+2)-m(k+1)^n\ln(k+1)\\
&+n(k+1)^{n-1}\ln(k+1)\Big)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I(n,m) &= (-1)^{n+m}\lim_{N\to\infty}\Big(\sum_{i=0}^{n}\sum_{j=1,j\neq m-i}^{m}\sum_{l=0}^{n-i}C_n^i C_{n-i}^l\frac{j(-1)^{m-l+i+j-1}}{i+j-m}\sum_{k=1}^{N}k^{n-l=J-M}\\
&-\sum_{i=0}^{n}\sum_{j=1,j\neq m-i}^{m}C_n^i\frac{j(-1)^{m+i+j-1}k^{n+j-m}}{i+j-m}+\sum_{i=0}^{\min\{m-1,n\}}C_n^i(m-i)\sum_{k=1}^{N}k^{n-i}\ln k\\
&-\sum_{i=0}^{\min\{m-1,n\}}C_n^i(m-i)\sum_{l=0}^{n-i}(-1)^l C_{n-i}^l\sum_{k=1}^{N}k^{n-i-l}\ln k\\
&+m\sum_{i=1}^{n}C_n^i\frac{(-1)^i}{i}\sum_{l=0}^{i}\sum_{k=1}^{N}k^{n-i+l}-m\sum_{i=1}^{n}C_n^i\frac{(-1)^i}{i}\sum_{k=1}^{N}k^{n}\\
&+\sum_{i=2}^{n}C_n^i\frac{(-1)^i}{i-1}\sum_{l=0}^{i-1}C_{i-1}^l\sum_{k=1}^{N}k^{n-i+l}-\sum_{i=2}^{n}C_n^i\frac{(-1)^i}{i-1}\sum_{k=1}^{N}k^{n-1}\\
&+\sum_{l=1}^{n-1}(-1)^l C_{n-1}^l\sum_{k=1}^{N}k^{n-2-l}+m\sum_{l=0}^{n}(-1)^l C_n^l\sum_{k=1}^{N}k^{n-l}\ln k+Q_{n,m}(N)\\
&-n\sum_{l=0}^{n-1}(-1)^l C_{n-1}^l\sum_{k=1}^{N}k^{n-l-1}\ln k-m\sum_{k=1}^{N}k^{n}\ln k+n\sum_{k=1}^{N}k^{n-1}\ln k+Pn,m,\\
&+n(k+1)^{n-1}\ln(k+1)\Big)
\end{aligned}
$$

where

$$
\begin{aligned}
P_{n,m} &= -m\sum_{i=0}^{n}C_n^i\sum_{j=1,j\neq m-i}^{m}\sum_{l=0}^{n-i}C_{n-i}^l\frac{j(-1)^{m-l+i+j-1}}{i+j-m}-m\sum_{i=1}^{n}\frac{(-1)^i}{i}C_n^i 2^i\\
&+m\sum_{i=1}^{n}\frac{(-1)^i}{i}C_n^i-\sum_{i=2}^{n}\frac{(-1)^i}{i-1}C_n^i 2^{i-1}+\sum_{i=2}^{n}\frac{(-1)^i}{i-1}C_n^i\\
&-\sum_{l=0}^{n-1}(-1)^l C_{n-1}^i 2^{n-2-l}-\sum_{l=0}^{n-1}(-1)^l C_{n-1}^i-m\sum_{l=0}^{n}(-1)^l C_n^i 2^{n-l}\ln 2\\
&+\sum_{l=0}^{n-1}(-1)^l C_{n-1}^i 2^{n-1-l}\ln 2,
\end{aligned}
$$

$$\begin{aligned} Q_{n,m}(N) &= \sum_{i=0}^{n} C_n^i \sum_{j=1, j\neq m-i}^{m} \frac{j(-1)^{m+i+j-1}}{i+j-m} N^{n+j-m} - \sum_{i=0}^{\min\{m-1,n\}} C_n^i (m-i) N^{n-i} \ln N \\ &+ \sum_{l=0}^{n-1} (-1)^l C_{n-1}^i (N+1)^{n-2-l} + m \sum_{l=0}^{n} (-1)^l C_n^l (N+1)^{n-l} \ln(N+1) \\ &- n \sum_{l=0}^{n-1} (-1)^l C_{n-1}^i (N+1)^{n-1-l} \ln(N+1). \end{aligned}$$

Noticing the identities

$$\begin{aligned} \sum_{i=0}^{n} \frac{(-1)^i}{\alpha+i} C_n^i &= \frac{n!}{\alpha(\alpha+1)...(\alpha+n)}, \text{ for } \alpha \neq 0, -1, ..., -n, \\ \sum_{j=1}^{n} (-1)^j j C_n^j a^{j-1} b^{n-j} &= -n(b-a)n-1, \\ \sum_{i=1}^{n} \frac{(-1)^i}{i} C_n^i a^i &= \sum_{i=1}^{n} \frac{(1-a)^i}{i} - H_n, \\ \sum_{i=2}^{n} \frac{(-1)^i}{i-1} C_n^i a^{i-1} &= nH_{n-1} - n \sum_{i=1}^{n-1} \frac{(1-a)^i}{i} - \frac{(1-a)^n - 1 - na}{a}. \end{aligned}$$

one gets

$$\begin{aligned} P_{n,m} &= \sum_{j=1, j\neq m-n}^{m} \frac{j(-1)^{m+n+j}}{n+j-m} - \frac{m(-1)^n}{n} + (n-m)\left(\ln 2 + \sum_{i=1}^{n-1} \frac{(-1)^i}{i}\right) \\ &+ \frac{(-1)^n}{2} - \delta_{n-1,0} \end{aligned} \tag{4.14}$$

$$Q_{n,m}(N) = \left(\begin{array}{c} \sum\limits_{i=0}^{n} C_n^i \sum\limits_{j=1, j\neq m-i}^{m} \frac{j(-1)^{m+i+j-1}}{i+j-m} N^{n+j-m} + mN^n \ln N - nN^{n-1} \ln N \\ - \sum\limits_{i=0}^{\min\{m-1,n\}} C_n^i (m-i) N^{n-i} \ln N - nN^{n-1} \ln(1+\frac{1}{N}) \\ + \frac{1}{N+1} N^{n-1} + mN^n \ln(1+\frac{1}{N}). \end{array} \right). \tag{4.15}$$

By the proof of Theorem 1 and noticing

$$\begin{aligned} N^k \ln(1+\frac{1}{N}) &= \sum_{i=1}^{k-1} \frac{(-1)^{i-1}}{i} N^{k-i} + \frac{(-1)^{k-1}}{k} + O(\frac{1}{N}), \\ \frac{1}{N+1} N^{n-1} &= N^{n-2} \sum_{i=i}^{n-2} \frac{(-1)^i}{N^i} + O(\frac{1}{N}), \end{aligned}$$

and $R(k) = 0$ for $k \geq 0$, we have

$$L(n,m) = \left(\begin{array}{c} \sum\limits_{i=0}^{n} C_n^i \sum\limits_{l=1}^{m-i-1} \sum\limits_{j=1,j\neq m-i}^{m-i-l} C_{n-i}^{l+n+j-m} \frac{j(-1)^{n+l+i-1}}{i+j-m} R(-l) + (-1)^{n-1}\gamma \\ -\sum\limits_{i=0}^{n} C_n^i \sum\limits_{j=1,j\neq m-i}^{m-n-1} \frac{j(-1)^{m+i+j-1}}{i+j-m} R(n-m+j) + \frac{n}{n-1}(-1)^{n-1} H(n-1) \\ + \sum\limits_{i=0}^{\min\{m-1,n\}} C_n^i (m-i) J(n-i) + (-1)^{n-2} H(n-1) + \frac{m}{n}(-1)^{n-1} \\ - \sum\limits_{i=0}^{\min\{m-1,n\}} C_n^i (m-i) \sum\limits_{l=0}^{n-i} (-1)^l C_{n-i}^l J(n-i-l) \\ +m\sum\limits_{l=1}^{n} (-1)^l C_n^l J(n-l) - n\sum\limits_{l=1}^{n-1} (-1)^l C_{n-1}^l J(n-1-l) + P_{n,m} \end{array} \right) \tag{4.16}$$

When $n = m$, (4.16) becomes (4.5). The proof is completed. ■

when $n = m = 1, 2, 3$,or 4, by (4.5) we obtain

$$\int_0^1 \{\frac{1}{x}\}\{\frac{1}{1-x}\}dx = 2\gamma - 1,$$

$$\int_0^1 \{\frac{1}{x}\}^2\{\frac{1}{1-x}\}^2 dx = 4\ln(2\pi) - 4\gamma - 5,$$

$$\int_0^1 \{\frac{1}{x}\}^3\{\frac{1}{1-x}\}^3 dx = 6\gamma + 2 - \varsigma(2) - 3\ln(2\pi) - \frac{18\varsigma'(2)}{\pi^2}$$

and

$$\int_0^1 \{\frac{1}{x}\}^3\{\frac{1}{1-x}\}^3 dx = \frac{2}{3}\varsigma(2) - \frac{2}{3}\varsigma(3) + \frac{24\varsigma'(2)}{\pi^2} + \frac{24\varsigma(3)}{\pi^2} + 4\ln(2\pi) - 8\gamma - \frac{11}{3}.$$

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