On Base for Generalized Topological Spaces

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Abstract

Let $X$ be a set. A subset $\mu$ of the power set $P(X)$ is called a generalized topology on $X$ if it contains $\emptyset$ and any union of elements of $\mu$ belongs to $\mu$. Therefore a generalized topology is not necessarily closed under finite intersections. In this paper, we first consider the properties of interior and closure. Next, we introduce the concept of frontier in a generalized topology and investigate some of its properties. We know that every subset of $P(X)$ is a subbase for a topology. The purpose of the present paper is to show that any subset of $P(X)$ is a base for a generalized topology.

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1 Preliminaries

The story of generalized topology goes back to 1963. That year, N. Levein in the paper “Semi-open Sets and Semi-continuity in Topological Spaces” tried to generalize a topology by replacing open sets with semi-open sets ([3]). After him, similar works have been done. For example, in 1965 the notion of $\alpha$-open sets ([6]), in 1979 feebly open sets ([4]), in 1982 pre-open sets ([5]) and in 1983 $\beta$-open sets ([1]) have been introduced. Finally in 1997, A. Császár generalized these new open sets by introducing the concept of $\gamma$-open sets ([2]). The concept of “generalized topology” was devised by him in 2002.
Recent years, many topologist have faced generalized topologies. Á. Császár is actively working on this subject, though he is in his 87. Most of the papers about generalized topologies are submitted to Acta Mathematica Hungarica journal whose Á. Császár is editor in chief.

We know that a subset $\tau$ of the power set is a called a topology on a set $X$ if

1. Any union of elements of $\tau$ belongs to $\tau$.
2. Any finite intersection of elements of $\tau$ belongs to $\tau$.
3. $\emptyset$ belongs to $\tau$.
4. $X$ belongs to $\tau$.

Of course, if we calculate the union of an empty subset of $\tau$, we gain $\emptyset$, therefore (3) is not necessary. In a generalized topological space, some of the above properties do not hold. The concept of a topological space is often generalized by replacing open sets with other kinds of subsets. In many cases, a generalized topology is a subset $\mu$ of $P(X)$ that contains $\emptyset$ and any union of elements of $\mu$ belongs to $\mu$. In other words, we replace the family of open sets with a larger one. Therefore every topology is a generalized topology. A set $X$ with a generalized topology $\mu$ on it, is called a generalized topological space and is denoted by $(X, \mu)$ or briefly by $X$, when no confusion can result. A generalized topology is named strong if $X \in \mu$.

The usual concepts defined in topological spaces can be used again in generalized topological spaces. A subset $B$ of $X$ is called $\mu$-open (or $\mu$-closed) if $B \in \mu$ (or $X - B \in \mu$). For $B \subseteq X$, let $I(B)$ be the largest $\mu$-open subset of $B$. Equivalently, $I(B)$ is the union of all $\mu$-open subsets of $B$. $I(B)$ is called the interior of $B$. Let $C(B)$ be the smallest $\mu$-closed subset which contains $B$. Equivalently, $C(B)$ is the intersection of all $\mu$-closed subsets which contain $B$. $C(B)$ is called the closure of $B$. A point $x \in X$ is called a $\mu$-cluster point of $B$ if $U \cap (B - \{x\}) \neq \emptyset$ for each $U \in \mu$ with $x \in U$. The set of all $\mu$-cluster points of $B$ is denoted by $d(B)$. The collection of all $\mu$-open sets that contain a point $x$ is denoted by $\mu_x$, namely

\[ \mu_x = \{U | U \in \mu, x \in U \}. \]

Theorems about the concepts interior, closure and cluster point remain true in generalized topological spaces.
Proposition 1.1. Let $B$ be a subset of a space $X$. Then the followings hold:

1. $I(B) \subseteq B \subseteq C(B)$.
2. $I(I(B)) = I(B)$ and $C(C(B)) = C(B)$.
3. If $B' \subseteq B$, then $I(B') \subseteq I(B)$ and $C(B') \subseteq C(B)$.
4. $I(B) = B$ iff $B$ is $\mu$-open.
5. $C(B) = B$ iff $B$ is $\mu$-closed.
6. $C(B) = X - I(X - B)$ and $I(B) = X - C(X - B)$.
7. $x \in C(B)$ iff $U \cap B \neq \emptyset$ for each $U \in \mu_x$.
8. $x \in I(B)$ iff $U \subseteq B$ for some $U \in \mu_x$.
9. $C(B) = B \cup d(B)$.
10. $x \notin d\{x\}$ for each $x \in X$.

Proof. See proposition 2.1 of [8].

Some relations that are true in topological spaces do not hold in generalized ones. For example the equations

$$I(B_1 \cap B_2) = I(B_1) \cap I(B_2), \quad C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$$

that are true in topological spaces do not remain true in generalized topological spaces ([8]). But now we want to know why these equations fail to hold. $I(B_1 \cap B_2) \subseteq I(B_1) \cap I(B_2)$, since $B_1 \cap B_2 \subseteq B_1$ and $B_1 \cap B_2 \subseteq B_2$. Why the reverse is not true? Let’s try to prove it! Let $x \in I(B_1) \cap I(B_2)$, then by definition, there exist $\mu$-open sets $U_1, U_2 \in \mu_x$ so that $x \in U_1 \subseteq B_1$ and $x \in U_2 \subseteq B_2$, therefore $x \in U_1 \cap U_2 \subseteq B_1 \cap B_2$. But $U_1 \cap U_2$ is not necessarily a $\mu$-open set, because $\mu$ is not necessarily closed under finite intersections. If we find a $U \in \mu$ so that $x \in U$ and it is contained in every $V \in \mu$ with $x \in V$ (In this case we call $x$ a representative element for $U$), then $U \subseteq U_1$ and $U \subseteq U_2$, therefore we will have:

$$x \in U \subseteq U_1 \cap U_2 \subseteq B_1 \cap B_2 \Rightarrow x \in I(B_1 \cap B_2).$$

Thus the equation $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$ will hold if every $x \in X$ is a representative element for some $U \in \mu_x$. Consequently the equation $C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$ will hold using De Morgan’s law and (6) of proposition 1.1. A space $X$ that every $x \in X$ is a representative element for some $U \in \mu_x$ is called a $C_0$-space in [8].
2 Frontier

We can also define frontier of a set $B \subseteq X$ in generalized topological spaces exactly like its definition in topological ones.

**Definition 2.1.** If $X$ is a generalized topological space and $B \subseteq X$, the frontier of $B$ is the set $\partial B = C(B) \cap C(X - B)$. The frontier of a set $B$ is also denoted by $Fr_X(B)$.

**Proposition 2.2.** For any subset $B$ of a generalized topological space $X$, we have:

1. $C(B) = B \cup \partial B$.
2. $C(B) = I(B) \cup \partial B$.
3. $I(B) = B - \partial B$.
4. $I(B) = C(B) - \partial B$.
5. $X = I(B) \cup B \cup I(X - B)$.

**Proof.** By frontier definition, we have:

(1)

$$B \cup \partial B = B \cup (C(B) \cap C(X - B))$$

$$= (B \cup C(B)) \cap (B \cup C(X - B))$$

$$= C(B) \cap X$$

$$= C(B).$$

(2)

$$I(B) \cup \partial B = I(B) \cup (C(B) \cap C(X - B))$$

$$= (I(B) \cup C(B)) \cap (I(B) \cup C(X - B))$$

$$= C(B) \cap (I(B) \cup (I(B))^c)$$

$$= C(B) \cap X$$

$$= X.$$

(3)

$$B - \partial B = B - (C(B) \cap C(X - B))$$

$$= (B - C(B)) \cup (B - C(X - B))$$

$$= B - C(X - B)$$

$$= B \cap (C(X - B))^c$$

$$= B \cap I(B)$$

$$= I(B).$$
\[ C(B) - \partial B = C(B) - (C(B) \cap C(X - B)) \]
\[ = C(B) - (C(B) \cap (I(B))^c) \]
\[ = C(B) \cap ((C(B))^c \cup I(B)) \]
\[ = (C(B) \cap (C(B))^c) \cup (C(B) \cap I(B)) \]
\[ = \emptyset \cup I(B) \]
\[ = I(B). \]

(5) Since \( X - I(B) = C(X - B) \), we have:
\[ X = (C(X - B))^c \cup C(X - B) = I(B) \cup C(X - B). \]

By frontier definition, it is clear that \( \partial B = \partial (X-B) \). Now if we use (1), we can write: \( C(X-B) = \partial B \cup (X-B) \). Therefore \( X = I(B) \cup \partial B \cup (X-B) \). In the last equation, \( I(B) \cup \partial B \cup (X-B) \supseteq I(B) \cup \partial B \cup (X-B) \) is clear. To prove the reverse, let \( x \in X - B \) and \( x \notin I(X - B) \), then \( x \in (I(X - B))^c = C(B) \), Therefore \( x \in C(B) = I(B) \cup \partial B. \)

### 3 Base

If \((X, \tau)\) is a topological space, a base for \(\tau\) is a collection \(\beta \subseteq \tau\) such that the set of all possible unions of subcollections \(\beta'\) of \(\beta\) equals \(\tau\). In other words, \(\tau = \{\cup \beta' | \beta' \subseteq \beta\}\). An arbitrary subset \(\beta\) of \(P(X)\) is not necessarily a base for some topology on \(X\). But we have the following proposition:

**Proposition 3.1.** \(\beta\) is a base for some topology on \(X\) iff

1. \(X = \cup_{B \in \beta} B\).
2. Whenever \(B_1, B_2 \in \beta\) with \(x \in B_1 \cap B_2\), there is some \(B_3 \in \beta\) with \(x \in B_3 \subseteq B_1 \cap B_2\).

**Proof.** See theorem 5.3 of [7].

**Definition 3.2.** If \((X, \tau)\) is a topological space, a subbase for \(\tau\) is a collection \(S \subseteq \tau\) such that the collection of all finite intersections of elements of \(S\) forms a base for \(\tau\).

**Proposition 3.3.** Any subset of \(P(X)\) is a subbase for some topology on \(X\).

**Proof.** See theorem 5.6 of [7].
Proof of proposition 3.3 states that by a given subset $S$ of $P(X)$, we can build a base $\beta$ if we make all finite intersections of elements of $S$ and then by making all possible unions of subcollections of $\beta$, we gain a topology $\tau$ on $X$.

The definition of base in generalized topological spaces is the same as topological ones.

**Definition 3.4.** Let $\beta \subseteq P(X)$, then $\beta$ is called a base for a generalized topology $\mu$ if $\mu = \{ \cup \beta' | \beta' \subseteq \beta \}$.

**Proposition 3.5.** $\beta \subseteq P(X)$ is a base for a generalized topology $\mu$ iff whenever $U$ is a $\mu$-open set and $x \in U$, then there exists $B \in \beta$ such that $x \in B \subseteq U$.

**Proof.** ($\Rightarrow$) Let $\beta$ be a base for $\mu$. Therefore $\mu = \{ \cup \beta' | \beta' \subseteq \beta \}$. If $U$ is a $\mu$-open set, then $U \in \mu$, thus there exists $\beta' \subseteq \beta$ such that $U = \cup \beta'$. Since $x \in U$, there is a $B' \in \beta'$ so that $x \in B'$, $B' \subseteq \cup \beta' = U$. Therefore $x \in B' \subseteq U$.

($\Leftarrow$) Let $U$ be a set, we show there exists a $\beta' \subseteq \beta$ so that $U = \cup \beta'$. By supposition for each $x \in U$ there is $B_x \in \beta$ in such away that $x \in B_x \subseteq U$. Now if we consider $\beta' = \{ B_x | x \in U \}$, it will be clear that $U = \cup \beta'$.

**Theorem 3.6.** $\beta$ is a base for some strong generalized topology iff $X = \cup_{B \in \beta} B$.

**Proof.** ($\Rightarrow$) Let $\beta$ be a base for a strong generalized topology $\mu$. Since $X$ is a $\mu$-open set, by proposition 3.5, for every $x \in X$ there exists a $B_x \in \beta$ so that $x \in B_x \subseteq U$. Therefore we have:

$$X \subseteq \cup_{x \in X} B_x \subseteq \cup_{B \in \beta} B \subseteq X \Rightarrow X = \cup_{B \in \beta} B.$$  

($\Leftarrow$) Suppose that $X = \cup_{B \in \beta} B$. Consider the set $\mu = \{ \cup \beta' | \beta' \subseteq \beta \}$. We show that $\mu$ is a strong generalized topology. If we take $\beta' = \emptyset$ then $\cup \beta' = \emptyset$, therefore $\emptyset \in \mu$. Any union of elements of $\mu$ again has a form like its elements and therefore belongs to $\mu$. On the other hand by definition 3.4 it is obvious that $\beta$ is a base for $\mu$.

**Theorem 3.7.** Any subset of $P(X)$ is a base for some generalized topology on $X$.

**Proof.** Let $\beta$ be a subset of $P(X)$. Consider the set $\mu = \{ \cup \beta' | \beta' \subseteq \beta \}$. If we repeat the proof of ($\Leftarrow$) in theorem 3.6, we conclude that $\mu$ is a generalized topology which has $\beta$ as a base.

Theorem 3.7 relates that with a given subset of $P(X)$, we can construct a generalized topology, namely the same work which we did with a subbase in a topology (See proposition 3.3).
References


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