A New Wide-Neighborhood Interior Point Algorithms For LCP

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Abstract

This paper establishes the polynomial convergence of a new class of path-following methods for linear complementarity problems (LCP). Namely, we show that the semilong-step path-following methods based on the $L_{\infty}$ seminorm neighborhood have the iteration-complexity bounds of $O(n)$.

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1 Introduction

LCPs arise in many areas, such as quadratic programming, bimatrix games, variational inequalities, and economic equilibria problems, and they have been the subject of much research interest. A number of direct as well as iterative methods have been proposed for their solution. The book by Cottle, Pang and Stone [1] is a good reference for pivoting methods developed to solve LCPs. Another important class of methods used to tackle LCPs are the interior-point and infeasible-interior-point methods, which were first designed to solve linear programs[2, 3, 4].
This paper considers path-following methods for LCP based on the Newton direction for the central path equation

\[ Xz = \mu e. \]

This Newton direction is quite natural in view of the fact that the neighborhoods of the central path used to develop polynomially convergent algorithms are all based on the \( L_2 \)-norm of the left-hand side of it\[^5\].

We consider one path-following LCP algorithm based on the above Newton direction, namely, a long-step path-following method based on the \( L_{\infty} \) semi-norm neighborhood, which is an extension of the infinity norm neighborhood for LP. We establish that algorithms (II) have iteration-complexity bounds of \( O(n) \), respectively, to get an \( \epsilon \)-complementary solution.

## 2 Preliminary Notes

Throughout this paper, uppercase roman letters denote matrices, lowercase roman letters denote vectors, and lowercase Greek letters denote scalars. We let \( R^n, R^n_+, R^n_{++} \) denote the set of \( n \)-dimensional vectors having real, nonnegative real, and positive real components, respectively. Also, we let \( R^{m \times n} \) denote the set of \( m \times n \) matrices with real entries. The set of all symmetric \( n \times n \) matrices is denoted by \( S^n \). \( S^n_+ \) denote the set of all matrices in \( S^n \) which are positive semidefinite. It is known that for each \( V \in S^n_+ \), there exists a unique \( U \in S^n_+ \), such that \( U^2 = V \). The matrix \( U \) is called the square root of \( V \) and is denoted by \( V^{1/2} \). For a vector whose \( i \)-th component is \( v_i \), we let \( \text{Diag}(v) \) denote the diagonal matrix whose \( i \)-th diagonal elements is \( v_i \) for every \( i = 1, \ldots, n \). \( v \in R^n \).

In addition, given vectors \( u \in R^m \) and \( v \in R^n \), we denote by \( (u, v) \) the vector \( (u^T, v^T)^T \in R^{m+n} \).

Certain matrices bear special notation, namely the matrices \( X, Z, \Delta X \). These matrices are the diagonal matrices corresponding to the vectors \( x, z, \Delta x \), respectively. The symbol 0 will be used to denote a scalar, vector, or matrices of all zeros; its dimensions should be clear from the context. Also, we denote by \( e \) the vector of all ones, and by \( I \) the identity matrix; their dimensions should be clear from the context.

For a vector \( z \in R^n \), \( \|z\|_2 = \sqrt{z^Tz} \) is the Euclidian norm, \( \|z\|_{\infty} = \max_{i=1,\ldots,n}|z_i| \) is the infinity norm.

### 2.1 The Newton direction and the generic algorithm

In this subsection, we derive the Newton direction for system (3), and state a generic path-following method on it. We end the subsection by give some existence results for this Newton direction.
It is now easy to see that the Newton direction \((\Delta x, \Delta z)\) for system (3) is the solution of the following system of linear equations:

\[
\begin{align*}
X\Delta z + Z\Delta x &= \gamma \mu e - Xz \\
M\Delta x - \Delta z &= 0.
\end{align*}
\] (1)

We next state the generic path-following feasible algorithm that will be studied in this paper.

**ALGORITHM I** Let \((x^0, z^0) \in F^0\), \(\mu_0 = (x^0)^Tz^0/n\) and set \(k = 0\).

**Repeat** until \((x^k)^Tz^k \leq \varepsilon\) do

**step1** Let \((x, z) = (x^k, z^k)\) and \(\mu = (x)^Tz/n\);

**step2** Choose a centrality parameter \(\gamma = \gamma_k \in [0, 1]\);

**step3** Compute the solution \((\Delta x, \Delta z)\) of system (4);

**step4** Choose a stepsize \(\theta_k > 0\) such that \((x^{k+1}, z^{k+1}) = (x^k, z^k) + \theta_k(\Delta x, \Delta z) \in \mathbb{R}^n_+\);

**step5** Set \(\mu_{k+1} = (x^{k+1})^Tz^{k+1}/n\) and increment \(k\) by 1.

**End** We end this section by stating the following straightforward results which will be used in section 3 to establish the polynomial convergence of Algorithm II, namely, the long step path-following algorithms.

**Lemma 2.1** For any vectors \(p, q > 0\), let the function \(f(\mu) = \|Pq - \mu e\|_2\), then \(\mu = \frac{p^Tq}{n}\) is the minimizer of \(f(\mu)\), where \(P = \text{diag}(p)\). Moreover, \(\|Pq - \frac{p^Tq}{n}\|_2 \leq \|Pq\|_2\).

**proof.**

\[
f(\mu)^2 = \|Pq - \mu\|_2^2 = \sum_{i=1}^n (p_iq_i - \mu)^2 = n\mu^2 - 2 \sum_{i=1}^n (p_iq_i)\mu + \sum_{i=1}^n p_i^2q_i^2.
\]

Observing the quadratic equation, the conclusion is obvious.

**Lemma 2.2** Let \(M \in S^n_+\), vectors \(x \in R^n_+\), \(z \in R^n_+\), then the matrix

\[
\begin{bmatrix}
Z & X \\
-M & I
\end{bmatrix}
\]

is nonsingular.

**proof.** Conversely, let the matrix above is singular, then there exists vector \((d^1, d^2) \in \mathbb{R}^{2n} \neq 0\), satisfies

\[
Zd^1 +Xd^2 = 0, -Md^1 + d^2 = 0.
\]

This implies

\[
X(X^{-1}Z + M)d^1 = 0.
\]

Considering the fact that \(X\), and \(X^{-1}Z + M\) are definite, we obtain \(d^1 = 0\) and \(d^2 = 0\). This is a contradiction with that \((d^1, d^2) \in \mathbb{R}^{2n} \neq 0\), then we complete the proof.

The proof of next lemma is straightforward and therefore we omit the details.
Lemma 2.3 For the algorithm above, the following equations hold

1. \( x(\theta)^T z(\theta) = (1 - \theta(1 - \gamma))x^T z + \theta^2 \Delta x^T \Delta z \);
2. \( X(\theta)z(\theta) - \frac{z(\theta)^T z(\theta)}{n} e = (1 - \theta)(Xz - \frac{x^T z}{n} e) + \theta^2 [\Delta X \Delta z - \frac{\Delta x^T \Delta z}{n} e] \), where \( x(\theta) = x + \theta \Delta x, z(\theta) = z + \theta \Delta z \).

3 Long-step path-following algorithm

In this section, we present a long step path-following algorithm whose iterates lie within a larger neighborhood of the central path:

\[ N^\infty_\leq(\eta) = \{(x, z) \in \mathcal{F}_0, \|Xz - \mu e\|_\infty \leq \eta \mu \} \]

where \( \mu \equiv x^T z/n \) and \( \eta \) is a constant such that \( \eta \in (0, 1) \).

ALGORITHM II

Let \( (x^0, z^0) \in N^\infty_\leq(\eta), \eta \in (0, 1), \gamma \in (0, 1), \) and \( \gamma \leq 2(1 - \eta) \), set \( k = 0 \).

Repeat until \( (x^k)^T z^k \leq \varepsilon \) do

step 1 Let \( (x, z) = (x^k, z^k) \), compute the solution \( (\Delta x, \Delta z) \) of system (4);

step 2 Compute \( \hat{\theta} = \max\{0 < \theta \leq 1 - \gamma|(x(\theta), z(\theta)) \in N^\infty_\leq(\eta), \forall \theta \in [0, \hat{\theta}]\}, \)

\( x(\theta) = x + \theta \Delta x, z(\theta) = z + \theta \Delta z \);

step 3 Set \( (x^{k+1}, z^{k+1}) = (x^k + \hat{\theta} \Delta x, z^k + \hat{\theta} \Delta z) \) and increment \( k \) by 1.

End

Lemma 3.1 Let \( \eta \in (0, 1), \gamma \in (0, 1), \gamma \leq 2(1 - \eta) \), and \( (x, z) \) is generated by Algorithm II, then \( \|r\|^2 \leq n \mu, r = (XZ)^{-1/2}(\gamma \mu e - Xz) \).

proof. In this case

\[
\|r\|^2 = \sum_{i=1}^{n} \frac{(\gamma \mu - x_i z_i)^2}{x_i z_i} \\
= \sum_{i=1}^{n} \left[ \frac{\gamma \mu^2}{x_i z_i} - 2 \gamma \mu + x_i z_i \right] \\
\leq \frac{n(\gamma \mu^2)}{(1 - \eta) \mu} - 2n \gamma \mu + n \mu \\
\leq n \mu.
\]

The last two relations hold because \( x_i z_i \geq (1 - \eta) \mu \) and \( \gamma \leq 2(1 - \eta) \), respectively.

Lemma 3.2 Let \( (\Delta x, \Delta z) \) be generated by Algorithm II, then the following inequality holds

\[ \Delta x^T \Delta z \leq x^T z/2. \]

proof. Define \( D = X^{-1/2}Z^{1/2}, \mu = x^T z/n \). Multiplying system on both sides by \( (XZ)^{-1/2} \), we have

\[
D^{-1} \Delta z + D \Delta x = (XZ)^{-1/2}(\gamma \mu e - Xz).
\]
Taking $L_2$-norm squared we obtain
\[ \| D^{-1} \Delta z \|_2^2 + \| D \Delta x \|_2^2 + 2 \Delta x^T \Delta z = \| (XZ)^{-1/2} (\gamma \mu e - Xz) \|_2^2. \]

Using $2 \Delta x^T \Delta z \leq \| D^{-1} \Delta z \|_2^2 + \| D \Delta x \|_2^2 + 2 \Delta x^T \Delta z$, we can easily conclude that
\[ 2 \Delta x^T \Delta z \leq \| (XZ)^{-1/2} (\gamma \mu e - Xz) \|_2^2 \leq n \mu. \]

Thus we complete the proof.

## 4 Main Results

In this section, we will establish the polynomial convergence of the algorithm.

**Lemma 4.1** With the notations above
\[ \mu(\hat{\theta}) \leq [1 - \hat{\theta}(1 - \gamma)] \mu, \]
where $\mu(\hat{\theta}) = \frac{x(z(\hat{\theta}))^T z(\hat{\theta})}{n}$.

**proof.** Using lemma 2.4(1), lemma 3.1 and considering the Algorithm II, we have
\[ \mu(\hat{\theta}) \leq [1 - \hat{\theta}(1 - \gamma) + \hat{\theta}^2/2] \mu \leq [1 - \hat{\theta}(1 - \gamma) + \hat{\theta}(1 - \gamma)/2] \mu = [1 - \frac{\hat{\theta}(1 - \gamma)}{2}] \mu, \]
which completes the proof.

By lemma 5.1, we can obtain that the improvement of the objective value depends on the size of $\hat{\theta}$, so we wish to bound $\hat{\theta}$ from below.

**Lemma 4.2** Let $\eta \in (0, 1), \gamma \in (0, 1), and \gamma \leq 2(1 - \eta)$, then we have $\hat{\theta} \geq \theta^*_\eta := \min\{1 - \gamma, \frac{\gamma \mu}{2\|\Delta x\Delta z\|_\infty}\}$ in Algorithm II.

**proof.** Suppose the Algorithm II,
\[ X(\theta) z(\theta) - \mu(\theta) e \geq (1 - \theta)[Xz - \mu e] + \theta^2[\Delta X \Delta z - \frac{\Delta x^T \Delta z \mu e}{n}] \]
\[ \geq -\{(1 - \theta)\|Xz - \mu e\|_\infty + 2\theta^2\|\Delta X \Delta z\|_\infty\} e \]
\[ \geq -\{1 - \theta + \theta \gamma\} \eta \mu e \]
\[ \geq -\eta \mu(\theta) e. \]

Thus we complete the proof.

**Lemma 4.3** Let vectors $p, q \in R^n$ such that $p^T q \geq 0$, then the following inequality holds
\[ \| Pq \|_2 \leq \frac{\sqrt{2}}{4} \| p + q \|_2^2. \]
**Lemma 4.4** With the notations above, we have \[ \| \Delta X \Delta z / \mu \|_\infty \leq \frac{\sqrt{2}}{4} n \] holds in Algorithm II.

**proof.** Define \[ D = X^{-1/2} Z^{1/2}, \mu = x^T z / n, \] from
\[ (D \Delta x)^T D^{-1} \Delta z = \Delta x^T \Delta z = \Delta x^T M \Delta x \geq 0 \]
and \[ \Delta X \Delta z = D \Delta X D^{-1} \Delta z, \] \[ D^{-1} \Delta z + D \Delta x = (XZ)^{-1/2}(\gamma \mu e - X z), \] then using the lemma 4.3, we have
\[ \| \Delta X \Delta z \|_\infty \leq \| \Delta X \Delta z \|_2 = \| D \Delta X D^{-1} \Delta z \|_2 \leq \frac{\sqrt{2}}{4} \| (XZ)^{-1/2}(\gamma \mu e - X z) \|_2^2 \leq \frac{\sqrt{2n \mu}}{4}. \]

**Theorem 4.5** Let \( \eta \in (0, 1) \), and \( \gamma \in (0, 1) \) be constants with \( \gamma \leq 2(1 - \eta) \). Then Algorithm II, will terminate in \( O(\log \varepsilon^{-1}) \) iterations.

**proof.** We first consider the Algorithm II. Using the lemma 4.1, we have
\[ \mu(\hat{\theta}) \leq [1 - \frac{\hat{\theta}(1 - \gamma)}{2}] \mu. \]
From lemma 4.2 and lemma 4.4, we conclude
\[ \hat{\theta} \geq \frac{\sqrt{2n \gamma}}{n}. \]
Moreover, we have
\[ \mu^{k+1} \leq \{1 - \frac{\sqrt{2n \gamma}}{2n} (1 - \gamma)\} \mu^k, \]
where \( \mu^k = (x^k)^T z^k / n \), which yields the results.

**References**


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