Notes on the Graded Radical of Graded Submodules

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Abstract

Let $G$ be a group with identity $e$, and let $R$ be a $G$-graded commutative ring, and let $M$ be a graded $R$-module. In general, the graded radical of a graded primary submodule is not graded prime. We study sufficient conditions for which this property holds in the modules setting.

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1 Introduction

Let $G$ be a group with identity $e$. A ring $(R, G)$ is called a $G$-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of $R$ such that $R = \bigoplus_{g \in G} R_g$ such that $1 \in R_e$ and $R_g R_h \subseteq R_{gh}$ for each $g$ and $h$ in $G$. For simplicity, we will denote the graded ring $(R, G)$ by $R$. If $R$ is $G$-graded, then an $R$-module $M$ is said to be $G$-graded if it has a direct sum decomposition $M = \bigoplus_{g \in G} M_g$ such that for all $g, h \in G$; $R_g M_h \subseteq M_{gh}$. An element of some $R_g$ or $M_g$ is said to be homogeneous element. A submodule $N \subseteq M$, where $M$ is $G$-graded, is called $G$-graded if $N = \bigoplus_{g \in G} (N \cap M_g)$ or if, equivalently, $N$
is generated by homogeneous elements. Moreover, $M/N$ becomes a $G$-graded module with $g$-component $(M/N)_{g} = (M_{g} + N)/N$ for $g \in G$. We write $h(R) = \bigcup_{g \in G} R_{g}$ and $h(M) = \bigcup_{g \in G} M_{g}$. A graded ideal $I$ of $R$ is said to be graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. A graded ideal $I$ of $R$ is said to be graded maximal if $I \neq R$ and if there is a graded ideal $J$ of $R$ such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$. A graded ring $R$ is called graded local if it has a unique graded maximal ideal. By a chain of graded prime ideals of a graded ring $R$ we mean a finite strictly increasing sequence $P_{1} \subseteq \ldots \subseteq P_{n}$; the graded dimension of this chain is $n$. We define the graded dimension of $R$ to be the supremum of the lengths of all chains of graded prime ideals in $R$, we denote by $Gdim(R)$. A proper graded submodule $N$ of a graded $R$-module $M$ is called graded prime submodule if $rm \in N$, then $m \in N$ or $r \in (N : M)$, where $r \in h(R), m \in h(M)$. A graded submodule $N$ of $R$-module $M$ is called graded maximal if $N \neq M$ and if there is a graded submodule $K$ of $M$ such that $N \subseteq K \subseteq M$, then $N = K$ or $K = M$.

2 The graded radical of graded submodules

Let $N$ be a graded submodule of a graded $R$-module $M$. The graded radical of $N$, denoted by $Grad(N)$, is defined to be the intersection of the graded prime submodules of $M$ containing $N$ if such exist, and $M$ otherwise. We say that a graded submodule $N$ is a graded radical submodule if $Grad(N) = N$.

Example 2.1 If $q$ is a graded primary ideal of graded ring $R$ then by [4, Lemma 1.8], $Grad(q)$ is a graded prime ideal of $R$. However, in the module case, $Q$ is a graded primary submodule does not necessarily imply that $Grad(Q)$ is a graded prime submodule. If $R = \mathbb{Z}[x]$ and $M$ is the graded $R$-module $R \oplus R$ with $N$ the graded submodule $R(2, x) + R(x, 0)$, then $N$ is a graded primary submodule of $M$ whose graded radical is not graded prime.

The following Lemma is known, but we write it here for the sake of references.

Lemma 2.2 Let $M$ be a graded module over a graded ring $R$. Then the following hold:
(i) If $I$ and $J$ are graded ideals of $R$, then $I + J$ and $I \cap J$ are graded ideals.
(ii) If $N$ is a graded submodule, $r \in h(R)$ and $x \in h(M)$, then $Rx$, $IN$ and $rN$ are graded submodules of $M$.
(iii) If $N$ and $K$ are graded submodules of $M$, then $N + K$ and $N \cap K$ are...
also graded submodules of $M$ and $(N : M)$ is a graded ideal of $R$.

(iv) Let $N_{\lambda}$ be a collection of graded submodules of $M$. Then $\sum_{\lambda} N_{\lambda}$ and $\bigcap_{\lambda} N_{\lambda}$ are graded submodues of $M$.

**Lemma 2.3** Let $Q$ be a graded primary submodule of a graded $R$-module $M$ such that $(Q : M)$ is a graded radical ideal of $R$, then $Q$ is a graded prime submodule.

**Proof.** Suppose that $r_g m_h \in Q$ with $m_h \not\in Q$ where $r_g \in h(R)$ and $m_h \in h(M)$. Then since $Q$ is graded primary, $r_g \in \text{Grad}(Q : M) = (Q : M)$. Therefore $r_g M \subseteq Q$, implying $Q$ is graded prime. □

**Theorem 2.4** Let $R$ be a graded domain with $\text{Gdim}(R) = 1$ and $M$ a graded $R$-module. Then for any graded primary submodule $Q$ of $M$, $\text{Grad}(Q)$ is a graded prime submodule.

**Proof.** Consider the graded ideal $(P : M)$ for any graded prime submodule $P$ containing $Q$. These ideals are graded prime by [1, Proposition 2.5], and $Q \subseteq P$ implies that $(Q : M) \subseteq (P : M)$. So we have $\text{Grad}(Q : M) \subseteq (P : M)$. For any one of these graded prime submodules $P$, we generate the chain of graded ideals $0 \subset \text{Grad}(Q : M) \subseteq (P : M)$. Since $\text{Gdim}(R) = 1$, we must have $\text{Grad}(Q : M) = (P : M)$ for any graded prime submodule $P$ containing $Q$. So $\bigcap_{Q \subseteq P} P = \text{Grad}(Q)$ is a graded prime submodule of $M$, because if $r_g m_h \in \text{Grad}(Q)$ with $m_h \not\in \text{Grad}(Q)$ where $r_g \in h(R)$ and $m_h \in h(M)$. Then there exists a graded prime submodule $Q \subseteq P$ such that $m_h \not\in P$ and $r_g m_h \in P$. Hence $r_g \in (P : M)$, thus $r_g \in \bigcap_{Q \subseteq P} (P : M) = (\bigcap_{Q \subseteq P} P : M)$ since $\text{Grad}(Q : M) = (P : M)$ for any $Q \subseteq P$. □

**Theorem 2.5** Let $(R, m)$ be a graded local ring and $M$ a graded $R$-module. Then any intersection of graded maximal submodules of $M$ is graded prime.

**Proof.** Let $\{M_i\}$ be a collection of graded maximal submodules of $M$. By [1], each $(M_i : M)$ is a graded maximal ideal of $R$ and so must be equal to $m$. Hence the intersection $\bigcap_{i \in I} M_i$ is a graded prime submodule by the proof of above Theorem. □

**Corollary 2.6** Let $(R, m)$ be a graded local ring, $M$ a graded $R$-module and $N$ a graded submodule of $M$ such that every graded prime submodule containing of $N$ is graded maximal. Then $\text{Grad}(N)$ is graded prime.

**Corollary 2.7** If $(R, m)$ is a graded local ring and $M$ a graded $R$-module, then every graded prime submodule of $M$ is graded maximal if and only if every graded radical submodule of $M$ is graded maximal.
Proof. Suppose that every graded prime submodule of $M$ is graded maximal and let $\text{Grad}(N)$ be a graded radical submodule of $M$. Then $\text{Grad}(N) = \bigcap_{Q \subseteq P} P$ is graded prime by Theorem 2.5, since each $P$ is graded maximal. By the hypothesis, $\text{Grad}(N)$ is graded maximal. The converse follows from the fact that if $P$ is a graded prime submodule, then $P = \text{Grad}(P)$. \qed

Definition 2.8 A graded prime submodule $P$ is said to be a minimal graded prime of a graded submodule $N$ if $N \subseteq P$ and if $P'$ is another graded prime submodule with $N \subseteq P' \subseteq P$, then $P = P'$.

A minimal graded prime of the 0 graded submodule is called a minimal graded prime of the graded module $M$.

Theorem 2.9 If $N$ is a graded submodule of graded $R$-module $M$, then $\text{Grad}(N)$ is the intersection of the minimal graded primes of $N$.

Proof. Let $P$ be a graded submodule of $M$ with $N \subseteq P$. Then $P/N$ is a graded prime submodule of $M/N$. Clearly every graded prime submodule has a minimal graded prime submodule, so there exists a graded prime submodule $P'$ of $M$ such that $P'/N$ is a minimal graded prime of $P/N$. So $P'/N \subseteq P/N$. Thus $N \subseteq P' \subseteq P$. If $P''$ is another graded prime submodule with $N \subseteq P'' \subseteq P'$, then we have $P'/N = P''/N$ so that $P' = P''$. Therefore $P'$ is a minimal graded prime of $N$. Thus $\text{Grad}(N) = \bigcap_{N \subseteq P} P = \bigcap_{N \subseteq P'} P'$ ($P'$ is minimal graded prime submodule of $N$). Hence the proof is complete. \qed

Definition 2.10 A graded submodule $N$ of a graded $R$-module $M$ has a reduced graded primary decomposition if there are finitely many graded primary submodules $Q_i$ such that $N = Q_1 \cap ... \cap Q_n$ and $Q_i$ does not contain $\nexists \bigcap_{j \neq i} Q_j$ for each $i$ and the graded prime ideals $\text{Grad}(Q_i : M)$ are distinct.

If $N = Q_1 \cap ... \cap Q_n$ is a reduced graded primary decomposition of graded submodule $N$, we will say that $\text{Grad}(Q_i)$ is an isolated graded prime submodule of $N$ if $\text{Grad}(Q_i)$ is minimal in the set $\{\text{Grad}(Q_1), ..., \text{Grad}(Q_n)\}$.

Theorem 2.11 Let $M$ be a graded $R$-module and $N$ a proper graded submodule of $M$. If $N$ has a reduced graded primary decomposition $N = Q_1 \cap ... \cap Q_n$ such that all the graded prime ideals associated with $N$ are isolated , then $(N : M) = (Q_1 : M) \cap ... \cap (Q_n : M)$ is a reduced graded primary decomposition of the graded ideal $(N : M)$.

Proof. Suppose not. Then since we are assuming the graded ideals $\text{Grad}(Q_i : M)$ are distinct, we must have $(Q_i : M) \nexists \bigcap_{j \neq i} (Q_j : M)$ for some $i$. Then $\text{Grad}(Q_i : M) \nexists \bigcap_{j \neq i} \text{Grad}(Q_j : M)$. So since $\text{Grad}(Q_i : M)$ is graded prime ideal, hence $\text{Grad}(Q_i : M) \supset \text{Grad}(Q_j : M)$ for some $j \neq i$ by [4, Proposition 1.4]. However, this final inclusion contradicts the assumption that $\text{Grad}(Q_i : M)$ is an isolated graded prime ideal of $N$. \qed
Corollary 2.12 Let $M$ be a graded $R$-module and $N$ a proper graded submodule of $M$. If $N$ has a reduced graded primary decomposition $N = Q_1 \cap \ldots \cap Q_n$ such that all the graded prime ideals associated with $N$ are isolated, then

(i) $N$ is graded primary if and only if $(N : M)$ is graded primary

(ii) $N$ is graded prime if and only if $(N : M)$ is graded prime.

Proof. The necessarily of each part is hold by [1, Proposition 1.5]. To show sufficiently, let $N = Q_1 \cap \ldots \cap Q_n$ be a reduced graded primary decomposition of $N$. By above Theorem $(N : M) = (Q_1 : M) \cap \ldots \cap (Q_n : M)$ is a reduced graded primary decomposition of the graded ideal $(N : M)$. If $(N : M)$ is graded primary, we must have $n = 1$ and so $N = Q_1$ is graded primary. If $(N : M)$ is a graded prime, then $N = Q_1$ is graded prime by Lemma 2.3.  

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References


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