The Generalized \( (G'/G) \)-Expansion Method for Solving Nonlinear Partial Differential Equations in Mathematical Physics

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Abstract

In this paper, we explore new applications of the generalized \( (G'/G) \)-expansion method and its algorithm are proposed by studying Wang's \( (G'/G) \)-expansion method and constructing a first order nonlinear ordinary differential equation with a third-degree nonlinear term to the nonlinear reaction-diffusion, the compound KdV-Burgers and the generalized shallow water wave equations where the balance numbers of which are positive integers. As results, some new exact traveling wave solutions are obtained which include solitary wave solutions.

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1 Introduction

In recent years, nonlinear wave equations have been played essential roles in many scientific and engineering areas such as fluid mechanics, plasma and elastic media and optical fibers, etc. Thus, it has had a considerable attention to
find explicit traveling wave solutions of those problems. Several methods have
been presented to obtain new exact solutions for many nonlinear evolution
equations such that the homogeneous balance method [26,41], the hyperbolic
tangent expansion method [31,42], the trial function method [14], the tanh-
function method [18, 20, 23,29], the theta function method [3-5], the nonlinear
transform method [10], the Hirota bilinear method [8,9], the Weierstrass elliptic
function method [21], the F-function expansion method [23-25], the inverse
scattering transform [1], the exp-function expansion method [7], the Jacobi elliptic
function expansion [4,13,15,30,35,40,41], the Backlund transform [19,22],
the generalized Riccati equation [33], the original (G'/G)-expansion method [27],
the sine-cosine method [32], the sub-ODE method [16, 28], the complex hyperbolic
function method [36], the truncated Painleve expansion [2], the rank analysis method [6],
the ansatz method [11,12], and so on.

The objective of this article is to use the generalized (G'/G)-expansion method
which proposed by Lü et al [17], to find the exact solutions of nonlinear evolu-
tion equations via the reaction-diffusion, the compound KdV-Burgers and
the generalized shallow water wave equations which play an important role in
mathematical physics. The main idea of this method is that the traveling wave
solutions of the nonlinear evolution equations can be expressed by polynomials
in G where G = G(ξ) and based on a first order nonlinear ordinary differential
equation G' = \( \sum_{i=0}^{3} h_i G^i \) with a third-degree nonlinear term, and \( t = \frac{d}{d\xi} \). The degree
of these polynomial can be determined by considering the homogeneous
balance between the highest order derivatives and the nonlinear terms appearing
in the given nonlinear equations. The coefficients of these polynomials can
be obtained by solving a set of algebraic equations resulted from the process
of using the proposed method. This method will play an important role in
expressing the traveling wave solutions in terms of hyperbolic, trigonometric
and the rational functions for the nonlinear evolution equations in mathemati-
cal physics. The generalized (G'/G)-expansion method used in this article can be
applied to further nonlinear equations as the difference-differential equations
which can be done in forthcoming articles.

2 Description of the generalized (G'/G)-expansion
method

Suppose that a nonlinear equation is given by

\[ F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2.1} \]

where \( u = u(x, t) \) is an unknown function, \( F \) is a polynomial in \( u = u(x, t) \)
The generalized $(\frac{G'}{G})$-expansion method

and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

The main steps of the generalized $(\frac{G'}{G})$-expansion method [17] are the following:

**Step 1.** The traveling wave variable

$$u(x, t) = u(\xi), \quad \xi = x - Vt, \quad (2.2)$$

where $V$ is a constant, permits us reducing Eq. (2.1) into an ODE in the form

$$P(u, u', u'', ...) = 0. \quad (2.3)$$

**Step 2.** Suppose that the solution of Eq. (2.3) can be expressed by a polynomial in $G$ as follows:

$$u(\xi) = \sum_{i=0}^{m} A_i \left( \frac{G'}{G} \right)^i, \quad (2.4)$$

where $G = G(\xi)$ is the solution of the first order nonlinear ODE in the form

$$G' = h_0 + h_1 G + h_2 G^2 + h_3 G^3, \quad (2.5)$$

where $A_i, h_0, h_1, h_2$ and $h_3$ are constants to be determined and $A_m \neq 0$, while $m$ is called the balance number.

**Step 3.** The positive integer $m$ can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.3) as follows: If we define the degree of $u(\xi)$ as $D[u(\xi)] = m$, then the degree of other expressions is defined by

$$D \left[ u^r \left( \frac{d^q u}{d\xi^q} \right)^s \right] = mr + s(q + m). \quad (2.6)$$

Therefore, we can get the value of $m$ in (2.4).

**Step 4.** Substituting (2.4) into (2.3) and using the ODE (2.5), collecting all terms with the same order of $G$ together, we get a polynomial in $G$ Equating each coefficient of this polynomial to zero, yields a set of algebraic equations, which can be solved to get $A_i$ and $V$. Since the general solution of Eq. (2.5) is well known to us, then substituting $A_i, V$ and the general solutions of (2.5) into (2.4), we have traveling wave solutions of Eq. (2.1).

**Remark 1** It is well known [39] that Eq. (2.5) admits the following special solutions:

**Theorem 1.** Suppose that $h_0 = 0, h_2 = 0, h_3 \neq 0$. 
(i) If \( h_1 \neq 0 \), then Eq. (2.5) has solutions \( G = \pm \sqrt{\frac{-h_3 - C_1 h_1 \exp(-2h_1 \xi) h_1}{-h_3 + C_1 h_1 \exp(-2h_1 \xi)}} h_1. \)

(ii) If \( h_1 = 0 \), then Eq. (2.5) has solution \( G = \pm \frac{1}{\sqrt{-2h_3 \xi + C_2}}. \)

**Theorem 2.** Suppose \( h_3 = 0 \).

(i) If \( h_0 \neq 0, h_1 \neq 0, h_2 \neq 0 \), that Eq. (2.5) has a solution
\[
G = -\frac{h_1}{2h_2} + \sqrt{\frac{4h_0 h_2 - h_1^2}{2h_2}} \tan \left( \frac{\xi + C_2}{h_2} \sqrt{4h_0 h_2 - h_1^2} \right).
\]

(ii) If \( h_0 \neq 0, h_1 \neq 0, h_2 = 0 \), then Eq. (2.5) has a solution
\[
G = \frac{h_0}{h_1} + C_4 \exp(h_1 \xi).
\]

(iii) If \( h_0 = 0, h_1 \neq 0, h_2 \neq 0 \), then Eq. (2.5) has a solution \( G = \frac{h_1}{-h_2 + C_5 h_1 \exp(-h_1 \xi)}. \)

(iv) If \( h_0 = 0, h_1 = 0, h_2 \neq 0 \), then Eq. (2.5) has a solution \( G = \frac{1}{-h_2 \xi + C_6}. \)

(v) If \( h_0 = 0, h_1 \neq 0 \), then Eq. (2.5) has a solution \( G = h_0 \xi + C_7. \)

where \( c_i (i = 1, ..., 7) \) are constants.

We are interested to record here the references [34, 37, 38, 39, 43, 44, 45] which are useful to the readers who are interesting to know more about the \((G'/G)\)-expansion method and its applications.

### 3 Applications

In this section, we apply the generalized \((G'/G)\)-expansion method to find new traveling wave solutions for some nonlinear PDEs in mathematical physics.

#### 3.1 Example 1. The nonlinear reaction-diffusion equation

In this section, we consider the following reaction-diffusion equation [39]

\[
 u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0, \quad (3.1)
\]

where \( \alpha, \beta \) and \( \gamma \) are nonzero constants.

The traveling wave variable (2.2) permits us converting Eq. (3.1) into the following ODE:

\[
 (\alpha + V^2)u'' + \beta u + \gamma u^3 = 0. \quad (3.2)
\]

Consider the homogeneous balance between \( u'' \) and \( u^3 \) in (3.2) we get \( m = 1 \).

From (2.4) we get

\[
 u(\xi) = A_0 + A_1 \left( \frac{G'}{G} \right), \quad A_1 \neq 0, \quad (3.3)
\]
where \( A_0, A_1 \) are constants to be determined and \( G = G(\xi) \) satisfies Eq. (2.5). It is easy to deduce that

\[
    u' = \left[ -A_1 h_0^2 \right] G^{-2} + \left[ -A_1 h_0 h_1 \right] G^{-1} + \left[ A_1 (h_1 h_2 + h_0 h_3) \right] G + \left[ A_1 \left( h_2^2 + 2h_1 h_3 \right) \right] G^2 
\]

\[
    + \left[ 3A_1 h_2 h_3 \right] G^3 + \left[ 2A_1 h_3^2 \right] G^4, \tag{3.4}
\]

\[
    u'' = \left[ 2A_1 h_0^3 \right] G^{-3} + \left[ 3A_1 h_0^2 h_1 \right] G^{-2} + \left[ A_1 h_0 \left( h_1^2 + 2h_0 h_2 \right) \right] G^{-1} + \left[ A_1 h_0 \left( 2h_1 h_2 + 3h_0 h_3 \right) \right] G^2 + \]

\[
    \left[ A_1 \left( 2h_2^3 + 14h_1 h_2 h_3 + 9h_0 h_3^2 \right) \right] G^3 + \left[ A_1 h_3 \left( 11h_2^2 + 12h_1 h_3 \right) \right] G^4 
\]

\[
    + \left[ 17A_1 h_2 h_3^2 \right] G^5 + \left[ 8A_1 h_3^3 \right] G^6. \tag{3.5}
\]

Substituting (3.3) and (3.5) into (3.2) we obtain a polynomial in \( G \). On equating the coefficients of the polynomial in \( G^i \) \((i = -3, \ldots, 6)\), to zero, we get a system of algebraic equations as follows:
where

\[ A_1(2(V^2 + \alpha) + \gamma A_1^2)h_0^3 = 0, \]

\[ 3A_1h_0^2(\gamma A_0A_1 + (V^2 + \alpha + \gamma A_1^2)h_1) = 0, \]

\[ A_1h_0(\beta + 3\gamma A_0^2 + 6\gamma A_0A_1h_1 + (V^2 + \alpha + 3\gamma A_1^2)h_1^2 + (2(V^2 + \alpha) + 3\gamma A_1^2)h_0h_2) = 0, \]

\[ \gamma A_0^3 + 3\gamma A_0^2A_1h_1 + A_0(\beta + 3\gamma A_1^2(h_1^2 + 2h_0h_2)) + A_1(\gamma A_1^2h_1^3 + h_1(\beta + 2(V^2 + \alpha + 3\gamma A_1^2)h_0h_2)) + 3(V^2 + \alpha + \gamma A_1^2)h_0h_2 = 0, \]

\[ A_1(h_2(\beta + 3\gamma A_0^2 + 6\gamma A_0A_1h_1 + (V^2 + \alpha + 3\gamma A_1^2)h_1^2 + (2(V^2 + \alpha) + 3\gamma A_1^2)h_0h_2) + 6h_0(\gamma A_0A_1 + (V^2 + \alpha + \gamma A_1^2)h_1)h_3) = 0, \]

\[ A_1(3(V^2 + \alpha + \gamma A_1^2)h_1h_2^2 + 3\gamma A_0^2h_3 + (\beta + (4(V^2 + \alpha) + 3\gamma A_1^2)h_1^2 + 2(5(V^2 + \alpha) + 3\gamma A_1^2)h_0h_2)h_3 + 3\gamma A_0A_1(h_2^2 + 2h_1h_3)) = 0, \]

\[ A_1((2(V^2 + \alpha) + \gamma A_1^2)h_2^3 + 2(7(V^2 + \alpha)h_1 + 3\gamma A_1(A_0 + A_1h_1))h_2h_3 + 3(3(V^2 + \alpha) + \gamma A_1^2)h_0h_3) = 0, \]

\[ A_1h_3((11(V^2 + \alpha) + 3\gamma A_1^2)h_2^2 + 3(4(V^2 + \alpha)h_1 + \gamma A_1(A_0 + A_1h_1))h_3) = 0, \]

\[ A_1(17(V^2 + \alpha) + 3\gamma A_1^2)h_2h_3^2 = 0, \]

\[ A_1 \left(8\left(V^2 + \alpha\right) + \gamma A_1^2\right)h_3^4 = 0. \] \hspace{1cm} (3.6)

On solving the above algebraic equations (3.6) by Mathematica, we obtain three families of solutions for Eq. (3.1) as follows:

\[ u_1 = \pm \sqrt{-\frac{\beta}{\gamma}} \pm \frac{k_1h_1^2\sqrt{-4\beta\gamma}Exp\left(-2h_1 \left(x \mp t\sqrt{\frac{\beta}{2h_1^3}} - \alpha\right)\right)}{\gamma h_1 \left(-h_3 + k_1h_1\sqrt{-4\beta\gamma}Exp\left(-2h_1 \left(x \mp t\sqrt{\frac{\beta}{2h_1^3}} - \alpha\right)\right)\right)}, \] \hspace{1cm} (3.7)

where \( h_1 \neq 0, h_3 \neq 0 \) and \( k_1 \) are arbitrary constants.
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\[ u_2 = \mp \sqrt{-\beta} \pm \frac{k_2 h_1^2 \sqrt{-4\beta\gamma} \exp \left(h_1 \left(x \mp t \sqrt{\frac{2\beta}{h_1^2}} - \alpha \right) \right)}{\gamma h_1 \left(-h_0 + k_2 h_1 \exp \left(h_1 \left(x \mp t \sqrt{\frac{2\beta}{h_1^2}} - \alpha \right) \right) \right)}, \quad (3.8) \]

where $h_1 \neq 0$, $h_0$ and $k_2$ are arbitrary constants.

\[ u_3 = \mp \sqrt{-\beta} \pm \frac{k_3 h_1^2 \sqrt{-4\beta\gamma} \exp \left(-h_1 \left(x \mp t \sqrt{\frac{2\beta}{h_1^2}} - \alpha \right) \right)}{\gamma h_1 \left(-h_2 + k_3 h_1 \exp \left(-h_1 \left(x \mp t \sqrt{\frac{2\beta}{h_1^2}} - \alpha \right) \right) \right)}, \quad (3.9) \]

where $h_1 \neq 0$, $h_2$ and $k_3$ are arbitrary constants.

### 3.2 Example 2. The nonlinear compound KdV-Burgers equation

In this section, we consider the following compound KdV-Burgers equation [39]

\[ u_t + \alpha uu_x + \beta u^2 u_x + \gamma u_{xx} - \delta u_{xxx} = 0, \quad (3.10) \]

where $\alpha, \beta, \gamma$ and $\delta$ are nonzero constants. This equation can be thought as the compound KdV-Burgers equation involving nonlinear dispersion and dissipation effects. The traveling wave variable (2.2) permits us converting Eq. (3.10) into the following ODE:

\[ -V u' + \alpha u u' + \beta u^2 u' + \gamma u'' - \delta u''' = 0. \quad (3.11) \]

Consider the homogeneous balance between $u'''$ and $u^2 u'$ in (3.11), we get $m = 1$. Using the same idea in Sec 3.1, we may choose the solution of Eq. (3.11) in the form

\[ u(\xi) = A_0 + A_1 \left( \frac{G'}{G} \right), \quad A_1 \neq 0, \quad (3.12) \]

where $A_0, A_1$ are constants to be determined and $G = G(\xi)$ satisfies Eq. (2.5). It is easy to deduce that

\[ u''' = [-6A_1 h_0^4]G^{-4} + [-12A_1 h_0^3 h_1]G^{-3} + [-A_1 h_0^2 (7h_1^2 + 8h_0 h_2)]G^{-2} + [-A_1 h_0 (h_1^3 + 8h_0 h_1 h_2 + 6h_0^2 h_3)]G^{-1} + [A_1 (h_1 h_2 (h_1^2 + 8h_0 h_2) + \]

\[ \]
Substituting (3.4), (3.5), (3.12), and (3.13) into Eq. (3.11), we obtain

\[ h_0 \left( 13h_1^2 + 18h_0h_2h_3 \right) G + [A_1(7h_1^2h_2^2 + 8h_1^3h_3 + 68h_0h_1h_2h_3 + h_0(8h_2^3 + 27h_0h_3^2))]G^2 + [3A_1(4h_1h_2^3 + h_2(17h_1^2 + 22h_0h_2)h_3 + 27h_0h_1h_3^2)]G^3 + [2A_1(3h_1^4 + 46h_1h_2^2h_3 + 2(14h_1^2 + 33h_0h_2)h_3^2)]G^4 + [25A_1h_3(2h_2^3 + 7h_1h_2h_3 + 3h_0h_3^2)]G^5 + [3A_1h_3^2(43h_2^2 + 32h_1h_3)]G^6 + [133A_1h_2h_3^3]G^7 + [48A_1h_3^4]G^8. \]  

Substituting (3.4), (3.5), (3.12), and (3.13) into Eq. (3.11), we obtain a polynomial in \( G \). On equating the coefficients of the polynomial in \( G \) \((i = -4, ..., 8)\), to zero, we get a system of algebraic equations, as follows:

\[ A_1(6\delta - \beta A_1^2)h_0^4 = 0, \]

\[ -A_1h_0^3(-2(\gamma + 6\delta h_1) + A_1(\alpha + 2\beta A_0 + 3\beta A_1h_1)) = 0, \]

\[ -A_1h_0^2(-V + \beta A_0^2 + (-3\gamma + 2\alpha A_1)h_1 + (-7\delta + 3\beta A_1^2)h_1^2 + A_0(\alpha + 4\beta A_1h_1) + 2(-4\delta + \beta A_1^2)h_0h_2) = 0, \]

\[ -A_1h_0((-\gamma + (\alpha + 2\beta A_0)A_1)h_1^2 + (-\delta + \beta A_1^2)h_1^3 + h_1(-V + \alpha A_0 + \beta A_0^2 + (-8\delta + 3\beta A_1^2)h_0h_2) + h_0((-2\gamma + (\alpha + 2\beta A_0)A_1)h_2 + (-6\delta + \beta A_1^2)h_0h_3)) = 0, \]

\[ \gamma A_1h_0(2h_1h_2 + 3h_0h_3) = 0, \]

\[ A_1(h_2(h_1(-V + \alpha A_0 + \beta A_0^2 + (\gamma + (\alpha + 2\beta A_0)A_1)h_1 + (-\delta + \beta A_1^2)h_1^2) + h_0(2(\gamma - 4\delta h_1) + A_1(\alpha + 2\beta A_0 + 3\beta A_1h_1))h_2) + h_0(-V + \beta A_0^2 + 2(3\gamma + \alpha A_1)h_1 + (-13\delta + 3\beta A_1^2)h_1^2 + A_0(\alpha + 4\beta A_1h_1) + 3(-6\delta + \beta A_1^2)h_0h_2)h_3) = 0, \]

\[ A_0(\alpha + 4\beta A_1h_1) + 3(-6\delta + \beta A_1^2)h_0h_2)h_3) = 0, \]

\[ A_0(\alpha + 4\beta A_1h_1) + 3(-6\delta + \beta A_1^2)h_0h_2)h_3) = 0, \]
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\[ A_1((-V + \alpha A_0 + \beta A_0^2 + (3\gamma + 2(\alpha + 2\beta A_0)A_1)h_1 + (-7\delta + 3\beta A_1^2)h_1^2)h_2^2 + +2(-4\delta + \beta A_1^2)h_0 h_2^2 + 2h_0(5\gamma - 34\delta h_1 + 2A_1(\alpha + 2\beta A_0 + +3\beta A_1 h_1))h_2 h_3 + h_3(2h_1(-V + \alpha A_0 + \beta A_0^2 + (2\gamma + (\alpha + 2\beta A_0)A_1)h_1 + +(-4\delta + \beta A_1^2)h_1^2) + 3(-9\delta + \beta A_1^2)h_1^2 h_3)) = 0, \]

\[ A_1((2(\gamma - 6\delta h_1) + A_1(\alpha + 2\beta A_0 + 3\beta A_1 h_1))h_1^2 + h_2(-3V + 3\beta A_0^2 + +2(7\gamma + 3\alpha A_1)h_1 + (-51\delta + 9\beta A_1^2)h_1^2 + 3A_0(\alpha + 4\beta A_1 h_1) + 3(-22\delta + +3\beta A_1^2)h_0 h_2)h_3 + 3h_0(3(\gamma - 9\delta h_1) + A_1(\alpha + 2\beta A_0 + 3\beta A_1 h_1))h_3^2) = 0, \]

\[ A_1((11\gamma - 92\delta h_1 + 4A_1(\alpha + 2\beta A_0 + 3\beta A_1 h_1)h_2^2 h_3 + +2(-V + \alpha A_0 + \beta A_0^2 + 2(3\gamma + (\alpha + 2\beta A_0)A_1)h_1 + +(-28\delta + 3\beta A_1^2)h_1^2)h_2^2 + 12(-11\delta + \beta A_1^2)h_0 h_2 h_3^2) = 0, \]

\[ A_1 h_3((-50\delta + 5\beta A_1^2)h_2^2 + (17\gamma - 175\delta h_1 + 5A_1(\alpha + +2\beta A_0 + 3\beta A_1 h_1))h_2 h_3 + 5(-15\delta + \beta A_1^2)h_0 h_2^2) = 0, \]

\[ A_1 h_3^2(-129\delta + 9\beta A_1^2)h_2^2 + 2(4(\gamma - 12\delta h_1) + +A_1(\alpha + 2\beta A_0 + 3\beta A_1 h_1))h_3) = 0, \]

\[ 7A_1(-19\delta + \beta A_1^2)h_2 h_3^2 = 0, \]

\[ 2A_1(-24\delta + \beta A_1^2)h_4^2 = 0. \] (3.14)

Consequently, we obtain seven families of solutions for Eq. (3.10) in the form

\[ u_1 = -3\alpha \sqrt{\beta} \mp \sqrt{6\beta} \gamma + 6\delta h_1) \]

\[ \frac{2k_1 h_1^2 \sqrt{6\delta} \beta \exp(-2h_1 \xi)}{\beta(-h_3 + k_1 h_1 \exp(-2h_1 \xi))}, \] (3.15)

where \(\xi = x - \frac{t}{12\beta \delta} \left(-3\alpha^2 \delta + 2\gamma^2 \beta + 24\delta^2 \beta h_1^2\right)\), and \(h_1 \neq 0, h_3 \neq 0\) and \(k_1\) are
arbitrary constants.

\[ u_2 = -\frac{3\alpha\sqrt{\delta} + \sqrt{6}\beta}{6\beta\sqrt{\delta}} + \frac{2\sqrt{6}\delta\beta h_3}{\beta \left(-2h_3 \left(x - \frac{t}{12\beta} \left(-3\alpha^2\delta + 2\gamma^2\beta\right) + k_2\right)\right), \quad (3.16) \]

\[ u_3 = \frac{-\alpha}{2\beta} + \frac{\gamma}{\sqrt{6}\beta\delta} - \frac{2\sqrt{6}\delta\beta h_3}{\beta \left(-2h_3 \left(x - \frac{t}{12\beta} \left(-3\alpha^2\delta + 2\gamma^2\beta\right) + k_3\right)\right), \quad (3.17) \]

where \( h_3 \neq 0, k_2 \) and \( k_3 \) are arbitrary constants.

\[ u_4 = \frac{-3\alpha\sqrt{\delta} \pm \sqrt{6}\beta(\gamma - 3\delta h_1)}{6\beta\sqrt{\delta}} \pm \frac{k_4h_1^2\sqrt{6}\beta E \exp(h_1\xi)}{\beta(-h_0 + k_4h_1\exp(h_1\xi))}, \quad (3.18) \]

where \( \xi = x - \frac{t}{12\beta} \left(-3\alpha^2\delta + 2\gamma^2\beta + 24\delta^2\beta h_1^2\right) \), and \( h_1 \neq 0 \) and \( k_4 \) are arbitrary constants.

\[ u_5 = \frac{-3\alpha\sqrt{\delta} \mp \sqrt{6}\beta(\gamma + 6\delta h_1)}{6\beta\sqrt{\delta}} \mp \frac{k_5h_1^2\sqrt{6}\beta E \exp(-h_1\xi)}{\beta(-h_2 + k_5h_1\exp(-h_1\xi))}, \quad (3.19) \]

where \( \xi = x - \frac{t}{12\beta} \left(-3\alpha^2\delta + 2\gamma^2\beta + 24\delta^2\beta h_1^2\right) \), and \( h_1 \neq 0 \) and \( k_5 \) are arbitrary constants.

\[ u_6 = \frac{-3\alpha\sqrt{\delta} + \sqrt{6}\beta\gamma}{6\beta\sqrt{\delta}} + \frac{h_2\sqrt{6}\delta\beta}{\beta(-h_2 \left(x - \frac{t}{12\beta} \left(-3\alpha^2\delta + 2\gamma^2\beta\right) + k_6\right)\right), \quad (3.20) \]

\[ u_7 = \frac{-\alpha}{2\beta} + \frac{\gamma}{\sqrt{6}\beta\delta} - \frac{h_2\sqrt{6}\delta\beta}{\beta(-h_2 \left(x - \frac{t}{12\beta} \left(-3\alpha^2\delta + 2\gamma^2\beta\right) + k_7\right)\right), \quad (3.21) \]

where \( h_2 \neq 0, k_6 \) and \( k_7 \) are arbitrary constants.

### 3.3 Example 3. The nonlinear generalized shallow water equation

In this section, we consider the following the generalized shallow water wave equation [39]

\[ u_{xxx} + \alpha u_x u_t + \beta u_t u_{xx} - u_{xt} - \gamma u_{xx} = 0, \quad (3.22) \]
The generalized \((G'/G)-\text{expansion method}\)

where \(\alpha, \beta\) and \(\gamma\) are nonzero constants. The traveling wave variable (2.2) permits us converting Eq. (3.22) into the following ODE:

\[-Vu^{(4)} - (\alpha + \beta)V u' u'' + (V - \gamma) u'' = 0.\]  
(3.23)

On integrating (3.23) with respect to \(\xi\) once, we get

\[C - Vu^{(3)} - \frac{1}{2}(\alpha + \beta)Vu'^2 + (V - \gamma)u' = 0,\]  
(3.24)

where \(C\) is a constant of integration. Consider the homogeneous balance between \(u'''\) and \(u^2\) in (3.24), we get \(m = 1\). We choose the solution of Eq. (3.24) in the form

\[u(\xi) = A_0 + A_1 \left( \frac{G'}{G} \right), \quad A_1 \neq 0,\]  
(3.25)

where \(A_0, A_1\) are constants to be determined and \(G = G(\xi)\) satisfies Eq. (2.5).

Substituting (3.4), (3.5), (3.13) and (3.25) into (3.24), we have a polynomial in \(G\). On equating the coefficients of the polynomial in \(G^i\) \((i = -4, ..., 8)\), to zero, we get a system of algebraic equations, as follows:

\[-\frac{1}{2}VA_1(-12 + (\alpha + \beta)A_1)h_0^4 = 0,\]
\[-VA_1(-12 + (\alpha + \beta)A_1)h_0^3h_1 = 0,\]
\[-\frac{1}{2}A_1h_0^2(V(-14 + (\alpha + \beta)A_1)h_1^2 - 2(-V + \gamma + 8Vh_0h_2)) = 0,\]
\[A_1h_0(Vh_1^3 + h_1(-V + \gamma + V(8 + (\alpha + \beta)A_1)h_0h_2) + V(6 + (\alpha + \beta)A_1)h_0^3h_3 = 0,\]
\[C + V(\alpha + \beta)A_1^2h_0(h_2(h_1^2 + h_0h_2) + 3h_0h_1h_3) = 0,\]
\[A_1(h_1h_2(V - \gamma - Vh_1^2 + V(-8 + (\alpha + \beta)A_1)h_0h_2) + h_0(V - \gamma + V(-13 + 2(\alpha + \beta)A_1)h_1^2 + 3V(-6 + (\alpha + \beta)A_1)h_0h_2)h_3) = 0,\]
\[\frac{1}{2}A_1(-h_2^2(V(14 + (\alpha + \beta)A_1)h_1^2 + 2(-V + \gamma + 8Vh_0h_2)) + 4h_1(V - \gamma - 4Vh_1^2 + V(-34 + (\alpha + \beta)A_1)h_0h_2)h_3 + 3V(-18 + (\alpha + \beta)A_1)h_0^2h_3^2) = 0,\]
\[-A_1(V(12 + (\alpha + \beta)A_1)h_1h_2^3 + h_2(-3V + 3\gamma + V(51 + 2(\alpha + \beta)A_1)h_1^2 + V(66 + (\alpha + \\
\beta)A_1)h_0h_2)h_3 + 81Vh_0h_1h_2^3) = 0,\]
\[-\frac{1}{2}A_1(V(12 + (\alpha + \beta)A_1)h_1^4 + 2V(92 + 5(\alpha + \beta)A_1)h_1^2h_3 + \\
2(-2V + 2\gamma + 2V(28 + (\alpha + \beta)A_1)h_1^2 + 3V(44 + (\alpha + \beta)A_1)h_0h_2)h_3^2) = 0,\]
\[-VA_1h_3((50 + 3(\alpha + \beta)A_1)h_3^3 + (175 + 8(\alpha + \beta)A_1)h_1h_2h_3 + (75 + 2(\alpha + \beta)A_1)h_0h_2^2) \\
-\frac{1}{2}VA_1h_3^2((258 + 13(\alpha + \beta)A_1)h_2^2 + 8(24 + (\alpha + \beta)A_1)h_1h_3) = 0,\]
\[-VA_1(133 + 6(\alpha + \beta)A_1)h_2h_3^2) = 0,\]
\[-2VA_1(24 + (\alpha + \beta)A_1)h_3^4 = 0.\]  

Consequently, we obtain four families of solutions for Eq. (3.22) in the form:

\[u_1 = A_0 - \frac{24k_1h_1^2\text{Exp} \left(-2h_1 \left(x - \frac{\gamma t}{1-4h_1^2}\right)\right)}{(\alpha + \beta) \left(-h_3 + k_1h_1\text{Exp} \left(-2h_1 \left(x - \frac{\gamma t}{1-4h_1^2}\right)\right)\right)},\]  

(3.27)

where \(h_1 \neq 0, h_3 \neq 0\) and \(k_1\) are arbitrary constants.

\[u_2 = A_0 - \frac{24h_3}{(\alpha + \beta)(-2h_3(x - \gamma t) + k_2)},\]  

(3.28)

where \(h_3 \neq 0\), and \(k_2\) are arbitrary constants.

\[u_3 = A_0 + \frac{12k_3h_1^2\text{Exp} \left(h_1 \left(x - \frac{\gamma t}{1-h_1^2}\right)\right)}{(\alpha + \beta) \left(-h_0 + k_3h_1\text{Exp} \left(h_1 \left(x - \frac{\gamma t}{1-h_1^2}\right)\right)\right)},\]  

(3.29)

where \(h_1 \neq 0, h_0\) and \(k_3\) are arbitrary constants.

\[u_4 = A_0 - \frac{3k_4\text{Exp} \left(-\frac{1}{2} \left(x - \frac{4\gamma t}{3}\right)\right)}{(\alpha + \beta) \left(-h_2 + k_4\text{Exp} \left(-\frac{1}{2} \left(x - \frac{4\gamma t}{3}\right)\right)\right)},\]  

(3.30)

where \(h_1 \neq 0, h_2\) and \(k_4\) are arbitrary constants.

**Remark 2** All Solutions presented in this article have been checked with Mathematica by putting them back into the original equations (3.1), (3.10), and (3.22).
4 Conclusions

In the present work, the generalized \((G')^k\)-expansion method has been successfully applied to find the exact solutions of the reaction-diffusion, the compound KdV-Burgers and the generalized shallow water wave equations. In this paper, we only have investigated the case when \(G' = \sum_{i=0}^{3} h_i G^i\). The proposed method in the future can be extended to the case when \(i \geq 4\).

References


The generalized \((G'/G)\)-expansion method


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