On Subclass of Close-to-Convex Functions

K. Al Shaqsi and *M. Darus

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
ommath@hotmail.com
*maslina@pkrisc.cc.ukm.my

Abstract

In this paper the authors studied the coefficient estimate of a class of functions starlike with respect to \( k \)-symmetric points defined by derivative operators \( D^n_\lambda \) introduced by Al-Shaqsi and Darus [6]. The integral representation and several coefficient inequalities of functions belonging to this class are obtained.

Mathematics Subject Classification: 30C45

Keywords: Univalent functions, starlike functions, close-to-convex, differential operator, \( k \)-symmetric points

1 Introduction

Let \( \mathcal{A} \) denote the class of functions of the form:

\[
f(z) = z + \sum_{m=2}^{\infty} a_m z^m,
\]

which are analytic in the unit disk \( U = \{ z : |z| < 1 \} \).

Also let \( \mathcal{S} \) denote the subclass of \( \mathcal{A} \) consisting of all functions which are univalent in \( U \). A function \( f \in \mathcal{A} \) is said to be starlike, denoted by \( S^* \) if \( \text{Re}\{ \frac{zf'(z)}{f(z)} \} > 0 \).

\(^1\)Corresponding author
The authors introduced the following differential operator (see [6]):

\[ D^0_\lambda f(z) = (1 - \lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda \geq 0, \quad (1.2) \]
\[ D^1_\lambda f(z) = (1 - \lambda) z f'(z) + \lambda z (z f'(z))', \quad (1.3) \]
\[ D^n_\lambda f(z) = D_\lambda \left( \frac{z(z^{n-1} f(z))^n}{n!} \right), \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \quad (1.4) \]

If the function \( f \) is given by (1.1), then we write

\[ D^n_\lambda f(z) = z + \sum_{m=2}^{\infty} [1 + \lambda(m - 1)] C(n, m) a_m z^m, \quad (1.5) \]

where

\[ C(n, m) = \binom{m + n - 1}{n} = \frac{\prod_{j=1}^{m-1} (j + n)}{(m - 1)!}, \quad (m \geq 2). \quad (1.6) \]

Sakaguchi [7] once introduced a class \( S^*_S \) of functions starlike with respect to symmetric points, which consists of functions \( f \in \mathcal{S} \) satisfying the inequality

\[ \text{Re} \left\{ \frac{z f'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in \mathbb{U}. \quad (1.7) \]

Several different authors had studied the class of Sakaguchi [7] and discussed this class and its subclasses (see [1,3-5, 8-13]). Chand and Singh [9] for example, had introduced a class \( S^*_S(k) \) of functions starlike with respect to \( k \)-symmetric points, which consists of functions \( f \in \mathcal{S} \) satisfying the inequality

\[ \text{Re} \left\{ \frac{z f'(z)}{f_k(z)} \right\} > 0, \quad z \in \mathbb{U}, \quad (1.8) \]

where

\[ f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z), \quad (k \geq 1; \varepsilon^k = 1). \quad (1.9) \]

In [6] the authors studied the class \( K_S^{(k)}(n, \lambda; \phi(z)) \) consisting of functions \( f \in \mathcal{A} \) for which

\[ \frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} < \phi(z), \quad z \in \mathbb{U}, \quad (1.10) \]
where $D^n_\lambda f(z)$ given by (1.4), and

$$D^n_k \in \lambda f_k(z) = D\lambda_n \left( \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} z^{n-1} f(\varepsilon^\nu z)^n \right)$$ (1.11)

where $k \geq 1$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$.

In this paper we define the class $K_{S}(^{(k)} n, \lambda, \alpha, \beta)$ consisting of analytic functions as the following:

**Definition 1.1** Let $f \in A$ and satisfies the following inequality:

$$\left|\frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} - 1\right| < \beta \left|\frac{\alpha z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} + 1\right|, \quad z \in U,$$ (1.12)

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $k \geq 1$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$.

We note that $K_{S}^{(2)}(0, 0, 1, 1) \equiv S_{S}^* [7]$, $K_{S}^{(2)}(0, 0, \alpha, \beta) \equiv S_{S}^{(2)}(\alpha, \beta) [11]$ and $K_{S}^{(k)}(0, 0, \alpha, \beta) \equiv S_{S}^{(k)}(\alpha, \beta) [1]$.

In fact, the class $K_{S}^{(k)}(n, \lambda, \alpha, \beta)$ is a special case of the class $K_{S}^{(k)}(n, \lambda; \phi)$ studied in [6] when $\phi(z) = (1 + \beta z)/(1 - \alpha \beta z)$.

**2 Coefficient Estimates**

First, we need a lemma of Lakshminarasimhan [10].

**Lemma 2.1** Let $H(z)$ be analytic in $U$ and satisfy the condition

$$\left|\frac{1 - H(z)}{1 + \alpha H(z)}\right| < \beta, \quad z \in U,$$ (2.1)

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, with $H(0) = 1$. Then we have

$$H(z) = \frac{1 - z\phi(z)}{1 + \alpha z\phi(z)},$$ (2.2)

where $\phi(z)$ is analytic in $U$ and $|\phi(z)| \leq \beta$ for $z \in U$. Conversely any function $H(z)$ given by (2.2) above is analytic in $U$ and satisfies (2.1).

Next we give the following lemma, which shall be used to obtain the coefficient estimates for functions in the class $K_{S}^{(k)}(n, \lambda, \alpha, \beta)$. 

Lemma 2.2 Let $f$ and $g$ belong to $A$ and satisfy

$$\left| \frac{z(D^n_\lambda f(z))'}{D^n_\lambda g(z)} - 1 \right| < \beta \left| \frac{\alpha z(D^n_\lambda f(z))'}{D^n_\lambda g(z)} + 1 \right|,$$

where $0 \leq \alpha \leq 1, 0 < \beta \leq 1, k \geq 1, \lambda \geq 0$ and $n \in \mathbb{N}_0$, with $f$ given by (1.1) and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$.

Then for $m \geq 2$,

$$\left( [1 + \lambda(m - 1)] C(n, m)(|ma_m - b_m|) \right)^2 \leq 2(\alpha \beta^2 + 1) \sum_{j=2}^{m-1} j [1 + \lambda(j - 1)] C(n, j)|a_j||b_j| \quad (|a_1| = |b_1| = 1).$$

(2.3)

Proof. We use the same method of Sudharsan et al [11]. By Lemma 2.1 we have

$$\frac{z(D^n_\lambda f(z))'}{D^n_\lambda g(z)} = \frac{1 - z\phi(z)}{1 + \alpha z\phi(z)}$$

(2.4)

where $\lambda \geq 0, n \in \mathbb{N}_0$, $\phi(z)$ is analytic in $U$ and $|\phi(z)| \leq \beta$ for $z \in U$.

Then

$$[\alpha z(D^n_\lambda f(z))' + D^n_\lambda g(z)] z\phi(z) = D^n_\lambda g(z) - z(D^n_\lambda f(z))'.$$

Now if

$$\psi(z) = z\phi(z) = \sum_{m=1}^{\infty} t_m z^m,$$

then

$$|\psi(z)| \leq \beta |z| \quad z \in U.$$

Therefore

$$\left( (\alpha + 1)z + \sum_{m=2}^{\infty} [1 + \lambda(m - 1)] C(n, m) [\alpha ma_m + b_m] z^m \right) \left( \sum_{m=1}^{\infty} t_m z^m \right)$$

$$= \sum_{m=2}^{\infty} [1 + \lambda(m - 1)] C(n, m) [b_m - ma_m] z^m.$$

(2.5)
Equating the coefficients of $z^m$ in (2.5), we have

$$
[1 + \lambda(m - 1)] \left[ b_m - ma_m \right] C(n, m)
= (\alpha + 1)t_m + [1 + \lambda] \left[ \alpha^2 a_2 + b_2 \right] C(n, 2)t_{m-2} + \cdots
+ [1 + \lambda(m - 2)] \left[ \alpha(m - 1)a_{m-1} + b_{m-1} \right] C(n, m - 1)t_1.
$$

Thus the coefficient combination on the right side of (2.5) depends only upon the coefficients combination

$$
[1 + \lambda] \left[ \alpha^2 a_2 + b_2 \right] C(n, 2) + \cdots + [1 + \lambda(m - 2)] \left[ \alpha(m - 1)a_{m-1} + b_{m-1} \right] C(n, m - 1)
$$
on the left side.

Hence for $m \geq 2$ we can write

$$
\left[ (\alpha + 1)z + \sum_{j=2}^{m-1} [1 + \lambda(j - 1)] C(n, j) \left[ \alpha ja_j + b_j \right] z^j \right] \psi(z)
= \sum_{j=2}^{m} [1 + \lambda(j - 1)] C(n, j) \left[ b_j - ja_j \right] z^j.
$$

(2.6)

Squaring the moduli of both sides of (2.6) and integrating along $|z| = r < 1$ and on using the fact that $|\psi(z)| \leq \beta|z|$, we obtain

$$
\sum_{j=2}^{m} \left[ 1 + \lambda(j - 1) \right] C(n, j) (|ja_j - b_j|^2) \left[ (\alpha + 1)^2 r^2 + \sum_{j=2}^{m-1} \left[ 1 + \lambda(j - 1) \right] C(n, j) (|\alpha ja_j + b_j|^2) r^{2j} \right] < \beta^2 (\alpha + 1)^2 r^2
$$

Letting $r \to 1$ on the last inequality, we obtain

$$
\sum_{j=2}^{m} \left[ 1 + \lambda(j - 1) \right] C(n, j) (|ja_j - b_j|^2) < \beta^2 (\alpha + 1)^2 + \sum_{j=2}^{m-1} \left[ 1 + \lambda(j - 1) \right] C(n, j) (|\alpha ja_j + b_j|^2) \left[ (\alpha + 1)^2 r^2 + \sum_{j=2}^{m-1} \left[ 1 + \lambda(j - 1) \right] C(n, j) (|\alpha ja_j + b_j|^2) r^{2j} \right]
$$
This implies that
\[
\left[ \left[ 1 + \lambda(m - 1) \right] C(n, m)(|ma_m - b_m|) \right]^2 \\
\leq \beta^2(1 + \alpha)^2 + \beta^2 \sum_{j=2}^{m-1} \left[ \left[ 1 + \lambda(j - 1) \right] C(n, j)(|ja_j + b_j|) \right]^2 \\
- \sum_{j=2}^{m-1} \left[ \left[ 1 + \lambda(j - 1) \right] C(n, j)(|ja_j - b_j|) \right]^2 \\
\leq \beta^2(1 + \alpha)^2 + (\beta^2 \alpha^2 - 1) \sum_{j=2}^{m-1} \left[ \left[ 1 + \lambda(j - 1) \right] C(n, j)(|ja_j + b_j|) \right]^2 j^2|a_j|^2 \\
+ (\beta^2 - 1) \sum_{j=2}^{m-1} \left[ \left[ 1 + \lambda(j - 1) \right] C(n, j)(|ja_j - b_j|) \right]^2 |b_j|^2 \\
+ 2\alpha \beta^2 \sum_{j=2}^{m-1} j \left[ 1 + \lambda(j - 1) \right] C(n, j)|a_j||b_j| + 2 \sum_{j=2}^{m-1} j \left[ 1 + \lambda(j - 1) \right] C(n, j)|a_j||b_j|.
\]

Then
\[
\left[ \left[ 1 + \lambda(m - 1) \right] C(n, m)(|ma_m - b_m|) \right]^2 \\
\leq 2\alpha \beta^2 \sum_{j=2}^{m-1} j \left[ 1 + \lambda(j - 1) \right] C(n, j)|a_j||b_j| + 2 \sum_{j=2}^{m-1} j \left[ 1 + \lambda(j - 1) \right] C(n, j)|a_j||b_j|,
\]

\(||a_1| = |b_1| = 1|), since 0 \leq \alpha \leq 1, 0 < \beta \leq 1, k \geq 1, \lambda \geq 0 and n \in \mathbb{N}_0.

First, we give two meaningful conclusions about the class \(K_S^{(k)}(n, \lambda, \alpha, \beta)\).

**Theorem 2.3** The function \(f \in K_S^{(k)}(n, \lambda, \alpha, \beta)\) if and only if
\[
\frac{z(D^n f(z))'}{D^n f_k(z)} \prec \frac{1 + \beta z}{1 - \alpha \beta z},
\]
where \(\prec\) denotes the subordination between analytic functions (see [2]).

**Proof.** Let \(f \in K_S^{(k)}(n, \lambda, \alpha, \beta)\). Then from (1.12) we have
\[
\left| \frac{z(D^n f(z))'}{D^n f_k(z)} - 1 \right|^2 < \beta^2 \left| \frac{\alpha z(D^n f(z))'}{D^n f_k(z)} + 1 \right|^2
\]
expanding it we get
\[
(1 - \alpha^2 \beta^2) \left| \frac{z(D^n f(z))'}{D^n f_k(z)} \right|^2 - 2(1 + \alpha \beta^2) \text{Re}\left\{ \frac{z(D^n f(z))'}{D^n f_k(z)} \right\} < \beta^2 - 1
\]
If $\alpha \neq 1$ or $\beta \neq 1$, we have
\[
\frac{\left| z(D^n_\lambda f(z))' \right|^2}{D^n_\lambda f_k(z)} - 2 \frac{1 + \alpha \beta^2}{1 - \alpha^2 \beta^2} \text{Re} \left\{ \frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} \right\} + \left( \frac{1 + \alpha \beta^2}{1 - \alpha^2 \beta^2} \right)^2 < \frac{\beta^2 - 1}{1 - \alpha^2 \beta^2} + \left( \frac{1 + \alpha \beta^2}{1 - \alpha^2 \beta^2} \right)^2,
\]
that is,
\[
\frac{\left| z(D^n_\lambda f(z))' \right|^2}{D^n_\lambda f_k(z)} - \frac{1 + \alpha \beta^2}{1 - \alpha^2 \beta^2} < \frac{\beta^2(1 + \alpha)^2}{(1 - \alpha^2 \beta^2)^2},
\]
then
\[
\frac{\left| z(D^n_\lambda f(z))' \right|^2}{D^n_\lambda f_k(z)} - \frac{1 + \alpha \beta^2}{1 - \alpha^2 \beta^2} < \frac{\beta(1 + \alpha)}{(1 - \alpha^2 \beta^2)^2}.
\]
That the value region of $G(z) = z(D^n_\lambda f(z))'/D^n_\lambda f_k(z)$ is contained in the disk whose center is $(1 + \alpha \beta^2)/(1 - \alpha^2 \beta^2)$ and radius is $\beta(1 + \alpha)/(1 - \alpha^2 \beta^2)$ maps the unit disk to the disk
\[
\left| w - \frac{1 + \alpha \beta^2}{1 - \alpha^2 \beta^2} \right| < \frac{\beta(1 + \alpha)}{(1 - \alpha^2 \beta^2)^2}.
\]
Note that $G(U) \subset p(U)$, $G(0) = p(0)$, and $p(z)$ is univalent in $U$, we obtain the conclusions
\[
\frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} < p(z) = \frac{1 + \beta z}{1 - \alpha z}.
\]
Conversely, let
\[
\frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} < \frac{1 + \beta z}{1 - \alpha \beta z},
\]
then
\[
\frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} = \frac{1 + \beta w(z)}{1 - \alpha \beta w(z)},
\]
where $w(z)$ is analytic in $U$, and $w(0) = 0$, $|w(z)| < 1$. By simple calculation we can easily obtain from (2.7) that
\[
\left| \frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} - 1 \right| < \beta \left| \frac{\alpha z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} + 1 \right|,
\]
that is, $f \in K^{(k)}_S(n, \lambda, \alpha, \beta)$.
If $\alpha = \beta = 1$, inequality (1.12) becomes
\[
\left| \frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} - 1 \right| < \left| \frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} + 1 \right|.
\]
It is clear that \((z(D^n_\lambda f(z)')/D^n_\lambda f_k(z)) < (1+z)/(1-z)\). The proof of Theorem 2.3 is complete.

**Remark 2.4** From Theorem 2.3 we have
\[
\text{Re}\left\{ \frac{z(D^n_\lambda f(z)')}{D^n_\lambda f_k(z)} \right\} > 0, \quad z \in U,
\]
(2.8)

Because of \(\text{Re}\left\{ (1+\beta z)/(1-\alpha \beta z) \right\} > 0\).

**Theorem 2.5** Let \(f \in K^{(k)}_S(n, \lambda, \alpha, \beta)\), then \(f_k \in S^*\).

**Proof.** Suppose that \(f \in K^{(k)}_S(n, \lambda, \alpha, \beta)\), substituting \(z\) by \(\varepsilon^\mu z\) in (2.8) respectively (\(\mu = 0, 1, 2, \ldots, k-1; \varepsilon^k = 1\)), then (2.8) is also true that is,
\[
\text{Re}\left\{ \frac{\varepsilon^\mu z(D^n_\lambda f(\varepsilon^\mu z)')}{\varepsilon^\mu D^n_\lambda f_k(\varepsilon^\mu z)} \right\} > 0, \quad (\mu = 0, 1, 2, \ldots, k-1).
\]
(2.9)

According to the definition of \(f_k(z)\) and \(\varepsilon^k = 1\), we know \(f_k(\varepsilon^\mu z) = \varepsilon^\mu f_k(z)\). Let \(\mu = 0, 1, 2, \ldots, k-1\) in (2.9) respectively, and obviously we can get
\[
\text{Re}\left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{\varepsilon^\mu z(D^n_\lambda f(\varepsilon^\mu z)')}{\varepsilon^\mu D^n_\lambda f_k(\varepsilon^\mu z)} \right\} = \text{Re}\left\{ \frac{z(D^n_\lambda f_k(z)')}{D^n_\lambda f_k(z)} \right\} > 0, \quad (2.10)
\]
that is \(f_k \in S^*\).

**Remark 2.6** From Theorem 2.5 and inequality (2.8), we know that if \(f \in K^{(k)}_S(n, \lambda, \alpha, \beta)\), then \(f\) is a close to convex function. So \(K^{(k)}_S(n, \lambda, \alpha, \beta)\) is a subclass of the class belongs to close-to-convex functions.

**Theorem 2.7** Let \(f \in K^{(k)}_S(n, \lambda, \alpha, \beta)\), then

(i) For \(l \geq 2\),
\[
\left[ 1 + \lambda k(l-1) \right] C(n, k(l-1) + 1)(l-1)k \right]^2 |a_{(l-1)k+1}|^2 
\leq 2(\alpha \beta^2 + 1) \sum_{\delta=1}^{l-1} ((\delta - 1)k + 1) \left[ 1 + \lambda k(\delta - 1) \right] C(n, k(i-1) + 1)|a_{k(\delta-1)+1}|^2
\]
(|\(a_1\) = 1).

(ii) For \(m \geq 2, m \neq (l-1)k + 1,\)
\[
\left[ 1 + \lambda (m-1) \right] C(n, m)m \right]^2 |a_m|^2
\]
\[ \frac{[m-2]+1}{k} \sum_{\delta=1}^{[m-2]+1} ((\delta - 1)k + 1)[1 + \lambda k(\delta - 1)] C(n, k(i-1) + 1)|a_{k(\delta-1)+1}|^2 \]

\[ (|a_1| = 1). \]

where \([m-2]+1\) denote the biggest integer \(\leq \frac{m-2}{k} + 1\).

**Proof.** By definition of \(f_k(z)\) we have

\[ f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z) \]

\[ = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} [\varepsilon^\nu z + \sum_{m=2}^{\infty} a_m (\varepsilon^\nu z)^m] \]

\[ = z + \sum_{l=2}^{\infty} a_{(l-1)k+1} z^{(l-1)k+1}. \]

and

\[ D^n f_k(z) = z + \sum_{l=2}^{\infty} [1 + \lambda k(l-1)] C(n, (l-1)k + 1)a_{(l-1)k+1} z^{(l-1)k+1}. \]

For \(f_k(z) \in S^*\), \(f(z)\) and \(f_k(z)\) satisfy the condition of Lemma 2.2. By using Lemma 2.2, let \(m = (l-1)k + 1\) in (2.3) we get (i) of Theorem 2.7. If \(m \neq (l-1)k + 1, m \geq 2\) from 2.3 we obtain (ii) of Theorem 2.7.

### 3 The Integral Representation

In this section, we give representation of functions belonging in the class \(K^{(k)}_S(n, \lambda, \alpha, \beta)\).

**Theorem 3.1** Let \(f \in K^{(k)}_S(n, \lambda, \alpha, \beta)\), then we have

\[ D^n f_k(z) = z \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{1 + \alpha \beta w(t)}{t[1 - \alpha \beta w(t)]} dt \right\}, \]

where \(D^n f_k(z)\) is defined by (1.11), \(w(z)\) is analytic in \(U\) and \(w(0) = 0, |w(z)| < 1\).

**Proof.** Suppose that \(f \in K^{(k)}_S(n, \lambda, \alpha, \beta)\), from Theorem 2.3 we have

\[ \frac{z(D^n f(z))'}{D^n f_k(z)} = \frac{1 + \beta w(z)}{1 - \alpha \beta w(z)}. \]
where \( w(z) \) is analytic in \( U \), and \( w(0) = 0 \), \( |w(z)| < 1 \). Substituting \( z \) by \( \varepsilon^\mu z \) in this equality respectively \( (\mu = 0, 1, 2, ..., k - 1; \varepsilon^k = 1) \), and using the same method in Theorem 2.5 we get
\[
\frac{z(D^n_\lambda f_k(z))'}{D^n_\lambda f_k(z)} = \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{1 + \beta w(\varepsilon^\mu z)}{1 - \alpha \beta w(\varepsilon^\mu z)}.
\]
then we get
\[
\frac{(D^n_\lambda f_k(z))'}{D^n_\lambda f_k(z)} - \frac{1}{z} = \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{(1 + \alpha)\beta w(\varepsilon^\mu z)}{z[1 - \alpha \beta w(\varepsilon^\mu z)]}.
\]
Integrating this equality we have
\[
\log \left\{ \frac{D^n_\lambda f_k(z)}{z} \right\} = \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^z \frac{(1 + \alpha)\beta w(\varepsilon^\mu \zeta)}{\zeta[1 - \alpha \beta w(\varepsilon^\mu \zeta)]} d\zeta = \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{(1 + \alpha)\beta w(t)}{t[1 - \alpha \beta w(t)]} dt,
\]
that is
\[
D^n_\lambda f_k(z) = z \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{(1 + \alpha)\beta w(t)}{t[1 - \alpha \beta w(t)]} dt \right\},
\]
the proof of Theorem 3.1 is complete.

**Theorem 3.2** Let \( f \in K_{S}^{(k)}(n, \lambda, \alpha, \beta) \), then we have
\[
D^n_\lambda f(z) = \int_0^z \frac{1 + \beta w(\zeta)}{1 - \alpha \beta w(\zeta)} \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu \zeta} \frac{(1 + \alpha)\beta w(t)}{t[1 - \alpha \beta w(t)]} dt \right\} d\zeta,
\]
where \( D^n_\lambda f(z) \) is defined by (1.4), \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \), \( |w(z)| < 1 \).

**Proof.** From Theorem 3.1 we have
\[
(D^n_\lambda f(z))' = \frac{D^n_\lambda f_k(z)}{z} \frac{1 + \beta w(z)}{1 - \alpha \beta w(z)}
\]
\[
= \frac{1 + \beta w(z)}{1 - \alpha \beta w(z)} \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{(1 + \alpha)\beta w(t)}{t[1 - \alpha \beta w(t)]} dt \right\},
\]
by integrating this equality we can obtain the conclusion of Theorem 3.2.
4 Sufficient Condition

At last, we give sufficient condition of functions belonging to the class $K^{(k)}_S(n, \lambda, \alpha, \beta)$.

**Theorem 4.1** Let the function $f$ be defined by (1.1). Then $f \in K^{(k)}_S(n, \lambda, \alpha, \beta)$ if and only if

$$
\sum_{m=1}^{\infty} [1 + \lambda(m - 1)]C(n, m)[(1 + \alpha \beta)(mk + 1) + \beta - 1]|a_{mk+1}|
+ \sum_{m=2}^{\infty} [1 + \lambda(m - 1)]C(n, m)(1 + \alpha \beta)|a_m| < (1 + \alpha)\beta
$$

where $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$.

**Proof.** Suppose that $f$ be defined by (1.1), then for $|z| = r < 1$ we have

$$
|z(D^n_{\lambda} f(z))' - D^n_{\lambda} f_k(z)| - \beta |\alpha z(D^n_{\lambda} f(z))' + D^n_{\lambda} f_k(z)|
= |z + \sum_{m=2}^{\infty} m[1 + \lambda(m - 1)]C(n, m)a_m z^m - z - \sum_{m=2}^{\infty} [1 + \lambda(m - 1)]C(n, m)a_m b_m z^m|
- \beta |\alpha z + \sum_{m=2}^{\infty} [1 + \lambda(m - 1)]C(n, m)a_m z^m + z + \sum_{m=2}^{\infty} [1 + \lambda(m - 1)]C(n, m)a_m b_m z^m|,
$$

where

$$
b_m = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(m-1)\nu}, \quad \varepsilon^k = 1. \quad (4.2)
$$

Thus we have

$$
|z(D^n_{\lambda} f(z))' - D^n_{\lambda} f_k(z)| - \beta |\alpha z(D^n_{\lambda} f(z))' + D^n_{\lambda} f_k(z)|
\leq |z + \sum_{m=2}^{\infty} [1 + \lambda(m - 1)]C(n, m)(m - b_m)|a_m|r_m - \beta((1 + \alpha)r
- \sum_{m=2}^{\infty} [1 + \lambda(m - 1)]C(n, m)(\alpha m + b_m)|a_m| r_m|
< r \left\{ -(1 + \alpha)\beta + \sum_{m=2}^{\infty} [1 + \lambda(m - 1)]C(n, m)((m - b_m) + \beta(\alpha m + b_m)C(n, m)|a_m| \right\}.
$$

From the definition of $b_m$ we know

$$
b_m = \begin{cases} 
1, & m = lk + 1, \\
0, & m \neq lk + 1,
\end{cases}
$$
substituting it into last inequality, we get
\[
\left|z(D^n_\lambda f(z))' - D^n_\lambda f_k(z)\right| - \beta \left|\alpha z(D^n_\lambda f(z))' + D^n_\lambda f_k(z)\right| \\
< \left\{ - (1 + \alpha)\beta + \sum_{m=1}^{\infty} [1 + \lambda(m - 1)]C(n, m)[(1 + \alpha\beta)(mk + 1) + \beta - 1]|a_{mk+1}| \\
+ \sum_{m=2}^{\infty} \sum_{m \neq lk + 1} [1 + \lambda(m - 1)]C(n, m)(1 + \alpha\beta)m|a_m| \right\} < 0.
\]

The proof of the theorem is complete.

ACKNOWLEDGEMENT: The work presented here was partially supported by SAGA: STGL-012-2006.

References


[6] K. Al Shaqsi and M. Darus, On univalent functions with respect to \(k\)-symmetric points defined by a generalized Ruscheweyh derivatives operator.(Submitted)


**Received: December 16, 2006**