On the Degree of Theta Pairs of Finite Groups¹

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Abstract

Let G be a finite group and M a maximal subgroup of G . A θ -pair of M is any pair of subgroups (C, D) of G such that (i) $D \triangleleft G$, $D < C$, (ii) $\langle M, C \rangle = G$, $\langle M, D \rangle = M$ and (iii) $\frac{C}{D}$ has no proper normal subgroup of $\frac{G}{D}$. A θ^* -pair of M is a pair of subgroups (C, D) satisfying conditions (i) and (iii) and a property that $D \leq M$ and $C^g \not\leq M$ for every $g \in G$.

In this paper, we introduce the degree of θ -pairs, denoted by $d\theta(G)$ as the ratio $|\theta(G)|/m(G)$, where $\theta(G)$ is the union of all θ -pairs of the maximal subgroups of G and $m(G)$ is the total number of distinct maximal subgroups of G . Similarly, we define the degrees of maximal θ -pairs, θ^* -pairs and maximal θ^* -pairs of a finite group G and give some evaluations on the above degrees for some simple groups, nilpotent groups and solvable groups. Moreover, we prove that if G is nilpotent then the degree of maximal θ -pairs of G is exactly 1.

Keywords: θ -pair, θ^* -pair, maximal θ -pair, maximal θ^* -pair, degree of θ -pair, degree of θ^* -pair

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1 Introduction

In 1990, Mukhrejee and Bhattacharya [4] introduced the notion of θ -pair for every maximal subgroup of a finite group. They have proved the existence of

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a θ -pair for any maximal subgroup M of a finite group G . Two years later, Beidleman and Smith [2], generalized the concept of θ -pair for infinite groups. In [8], Yaoqing improved the definition of θ -pair for a maximal subgroup of a finite group G to a maximal θ -pair and normal θ -pair. He proved that for any maximal subgroup M of a finite group G , there exists a normal maximal θ -pair related to M . The investigation on θ -pair are continued by others in [1,7-9]. Most of this research deals with the structure of finite groups. Later, in 2000, Shirong and Yaoqing in [6], defined the new concept θ^* -pair which is a special case of θ -pairs by some additional conditions. Using the concept of θ^* -pair, they obtained several results on maximal θ^* -pairs which imply a finite group G to be solvable or supersolvable.

In this paper, we introduced a degree on the θ -pairs of a finite group G which is called θ -pairs degree of G and will be stated later. In this article, we prove that if G is a finite nilpotent group, then the θ -pairs and θ^* -pairs degrees are exactly 1. Moreover, we will compare the above degrees of a group G with the quotient group $\frac{G}{N}$, where N is a normal subgroup of G and contained in the Frattini subgroup $\Phi(G)$. Now we give some definitions and basic results.

2 Some definitions and basic results

All groups considered in this paper, are assumed to be finite. We also denote $M \leq_{\max} G$ as a convenience that M is a maximal subgroup of G . Now, we start with the definition of θ -pair given by Mukhrejee and Bhattacharya in [4].

Definition 2.1 Let G be a finite group and M a maximal subgroup G . A θ -pair of M is any pair (C, D) of subgroups satisfying the following conditions :

- (a) $D \triangleleft G, D < C$.
- (b) $C < M, C \geq G$ and $C < M, D \geq M$.
- (c) $\frac{C}{D}$ has no proper normal subgroup of $\frac{G}{D}$.

Definition 2.2 Let (C, D) be a θ -pair of a maximal subgroup M of G . Then (C, D) is called a normal θ -pair if $C \triangleleft G$ and is said to be a maximal θ -pair if there is no θ -pair (C', D') such that $C < C'$.

The set of all θ -pairs of a maximal subgroup M of G is denoted by $\theta(M)$. We also denote $\theta(G)$ as the union of $\theta(M)$ for all maximal subgroups M of G . Similarly, $\theta_m(M)$ and $\theta_m(G)$ will be denoted for those of θ -pairs which are maximal.

It is clear that for every maximal subgroup M of G , $\theta(M)$ and $\theta_m(M)$ are not empty. Since (C, M_G) is a θ -pair of M , where M_G is the core of M in G and C is a subgroup of G such that $\frac{C}{M_G}$ is a chief factor of G . It is easy to see that (C, M_G) is a maximal θ -pair of M as well.

Definition 2.3 given a maximal subgroup M of a finite group G , a θ^* -pair of M is any pair (C, D) of subgroup of G such that

- (a) $D < C$ and $D \triangleleft G$,
- (b) $D \leq M$ but $C^g \not\leq M$ for every $g \in G$ and
- (c) $\frac{C}{D}$ has no proper normal subgroup of $\frac{G}{D}$.

Similar to those of θ -pairs, we can define the concepts : normal, maximal θ^* -pair and the notations $\theta^*(M)$, $\theta_m^*(M)$ and $\theta^*(G)$.

By the above two definitions, we can see that every θ^* -pair of M is a θ -pair of M and so $\theta^*(M) \subseteq \theta(M)$. Thus, we have $\theta^*(G) \subseteq \theta(G)$. It is also clear that a normal θ^* -pair is certainly maximal and $\theta^*\text{-pair}(M^g) = \theta^*\text{-pair}(M)$ for every maximal subgroup M of G and every $g \in G$.

In the rest of this section, we state some known results on the θ -pairs and θ^* -pairs. We omitt the proofs and one may find them in [4].

Theorem 2.4 If (C, D) is a maximal θ -pair in $\theta(M)$ and $N \triangleleft G$, $N < D$, then $(\frac{C}{N}, \frac{D}{N})$ is a maximal θ -pair in $\theta(\frac{M}{N})$. Conversely, if $(\frac{C}{N}, \frac{D}{N})$ is a maximal θ -pair in $\theta(\frac{M}{N})$, then (C, D) is a maximal θ -pair in $\theta(M)$.

Theorem 2.5 Let G be a finite group. Then G is nilpotent if and only if $\theta(M)$, for every $M \leq_{\max} G$, contains a maximal pair (C, D) such that $\frac{G}{D}$ is nilpotent.

Theorem 2.6 A solvable group G is nilpotent if and only if for all $M \leq_{\max} G$, $\theta(M)$ contains exactly one maximal pair.

There are some similar results on θ^* -pair that we referred to [5] and [6].

3 Degree of θ -pairs and θ^* -pairs

In this section, we introduce a concept of degree for θ -pairs and θ^* -pairs of a finite group G . We state the following definition.

Definition 3.1 Let G be a finite group and $m(G)$ be the total number of distinct maximal subgroup of G . Then, $d\theta(G)$ is said to be the degree of θ -pairs of G and is

$$d\theta(G) = \frac{|\theta(G)|}{m(G)}.$$

Similarly, the degree of θ^* -pair of G is denoted by

$$d\theta^*(G) = \frac{|\theta^*(G)|}{m(G)}.$$

We may also define the above degree for maximal θ -pairs and θ^* -pairs of G . We denote them by $d\theta_m(G)$ and $d\theta_m^*(G)$, respectively. In the following lemma we compare the above degrees.

Lemma 3.2 Let G be a finite group G . then

- (i) $d\theta_m(G) \leq d\theta(G)$;
- (ii) $d\theta_m^*(G) \leq d\theta^*(G)$;
- (iii) $d\theta_m^*(G) \leq d\theta_m(G)$.

Proof It is clear by definitions 2.2 and 2.3.

Lemma 3.3 Let G be a finite non-abelian simple group. Then $|\theta_m(G)| = 1$.

Proof It is clear that $(G, 1)$ is a θ -pair for every maximal subgroup M of G . So we have to show that $(G, 1)$ is a unique maximal θ -pair. If (C, D) is a θ -pair of a maximal subgroup of G , then there are two possibilities for D . If $D = G$, then by condition (b) of Definition 2.1, we should have $\langle M, D \rangle = M$ which is a contradiction. Thus, D must be the identity subgroup. So, $(C, D) = (C, 1)$ and therefore

$$\theta(M) = \{(C, 1) : \langle C, M \rangle = G \text{ and } C \text{ has no proper normal subgroup of } G\}.$$

Since $(G, 1) \in \theta(M)$ and is the maximal element of $\theta(M)$, we have $\theta_m(G) = \{(G, 1)\}$ and the proof is completed.

Lemma 3.4 Let G be a finite non-abelian simple group. Then $m(G) > 1$.

Proof Assume that $m(G) = 1$ and M is a unique maximal subgroup of G . Then we should have $M^g = M$ for every $g \in G$ and so $M \triangleleft G$. If $M = 1$, then G is a cyclic group of prime order which is a contradiction. Hence $M \neq 1$ and it implies that $m(G) > 1$.

Theorem 3.5 Let G be a finite non-abelian simple group. Then

$$d\theta_m(G) = d\theta_m^*(G) = \frac{1}{m(G)} < 1.$$

Proof By lemmas 3.2 and 3.3, the proof is clear.

The degree of θ -pairs or θ^* -pairs may have the values equal or bigger than 1. The following example proves this fact.

Example 3.6

(i) Let $G = S_3$, then we have 4 maximal subgroups

$$M_1 = \langle (12) \rangle, M_2 = \langle (13) \rangle, M_3 = \langle (23) \rangle$$

and $M_4 = A_3$. We can easily see that

$$\begin{aligned}\theta(G) = \theta^*(G) &= \theta_m(G) = \theta_m^*(G) = \{(G, A_3), (A_3, 1), \\ &\quad (M_1, 1), (M_2, 1), (M_3, 1)\}.\end{aligned}$$

Thus

$$d\theta(G) = d\theta^*(G) = d\theta_m(G) = d\theta_m^*(G) = \frac{5}{4} > 1.$$

(ii) Let $G = \mathbb{Z}_3$, then by the similar method as the previous example we can find that $d\theta_m(G) = d\theta_m^*(G) = 1$.

Theorem 3.7 Let G be a cyclic group of prime order. Then

$$d\theta(G) = d\theta^*(G) = d\theta_m(G) = d\theta_m^*(G) = 1$$

Proof It is clear that the only proper subgroup of G is the identity subgroup. Thus $m(G) = 1$ and if $M = 1$ is a maximal subgroup of G , then $(G, 1)$ is the only θ -pair and, of course is a θ^* -pair and maximal. Therefore $\theta(M) = \theta_m(M) = \theta_m^*(M) = \{(G, 1)\}$ and it completes the proof.

The following theorem is a comparison of the maximal θ -pair degree of group G and quotient group $\frac{G}{N}$.

Theorem 3.8 Let G be a nilpotent group and N be a normal subgroup of G such that $N \subseteq \Phi(G)$. Then $d\theta_m(G) = d\theta_m(\frac{G}{N})$.

Proof First, we prove that $m(G) = m(\frac{G}{N})$. If M is a maximal subgroup of G , then $\frac{M}{N}$ is a maximal subgroup of $\frac{G}{N}$, because if there exists a subgroup $\frac{H}{N}$ of $\frac{G}{N}$ such that $\frac{M}{N} \subsetneq \frac{H}{N} \subsetneq \frac{G}{N}$, then $M \subsetneq H \subsetneq G$ which is a contradiction. Thus $\frac{M}{N}$ is a maximal subgroup of $\frac{G}{N}$. Conversely, if $\frac{M}{N}$ is a maximal subgroup of $\frac{G}{N}$, then by the similar method, M is a maximal subgroup of G . Therefore, we have $m(G) = m(\frac{G}{N})$. Now, assume that M is a maximal subgroup of G . Then $M \triangleleft G$ and so (G, M) is a θ -pair of M in G , because $\frac{G}{M}$ is simple. We can also see that (G, M) is a maximal θ -pair of M . Since, if (C, D) is another maximal θ -pair of M , then $C = G$ and so (G, D) is a normal maximal θ -pair. Hence, by (Theorem 2.3, [5]), $D = M_G$ and therefore $(C, D) = (G, M)$. It implies that $\theta_m(M) = \{(G, M)\}$. Now, assume that $m(G) = k$ and $M_1, M_2, \dots, M_k$ are distinct maximal subgroups of G , then

$$\theta_m(G) = \{(G, M_1), (G, M_2), \dots, (G, M_k)\}.$$

By the similar way we can show that $\theta_m(\frac{M}{N}) = \{(\frac{G}{N}, \frac{M}{N})\}$ and so

$$\theta_m(\frac{G}{N}) = \{(\frac{G}{N}, \frac{M_1}{N}), (\frac{G}{N}, \frac{M_2}{N}), \dots, (\frac{G}{N}, \frac{M_k}{N})\}.$$

Hence

$$d\theta_m(G) = \frac{|\theta_m(G)|}{m(G)} = \frac{|\theta_m(G)|}{m(\frac{G}{N})} = d\theta_m(\frac{G}{N}).$$

Proposition 3.9 Let G be a nilpotent group. Then $d\theta_m(G) = 1$.

Proof it is clear by Theorem 3.8.

We can establish the above result for the degree of θ^* -pairs and the proofs are very similar.

Finally, we state the following conjectures that we may find some evidences on it.

Conjecture 1 If $d\theta_m(G) = 1$, then G is nilpotent.

Conjecture 2 Let G_1 and G_2 be two finite groups. Then

$$d\theta_m(G_1 \times G_2) \leq d\theta_m(G_1)d\theta_m(G_2).$$

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